Discrete Mathematics (2009 Spring)
Graphs (Chapter 9, 5 hours)

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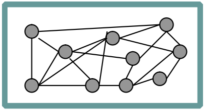
What are Graphs?

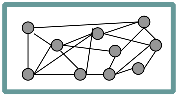
- General meaning in everyday math: A plot or chart of numerical data using a coordinate system.

- Technical meaning in discrete mathematics: A particular class of discrete structures (to be defined) that is useful for representing relations and has a convenient webby-looking graphical representation.



Simple Graphs

- Correspond to symmetric binary relations R.



Visual Representation of a Simple Graph

- A simple graph $G = (V, E)$ consists of:
  1. A set V of vertices or nodes (V corresponds to the universe of the relation R).
  2. A set E of edges / arcs / links: unordered pairs of [distinct?] elements $u, v \in V$, such that uRv.

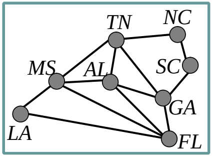
Example of a Simple Graph

- Let V be the set of states in the far-southeastern U.S.:

$$V = \{\text{FL,GA,AL,MS,LA,SC,TN,NC}\}.$$

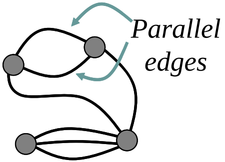
- Let

$$\begin{aligned} E &= \{\{u, v\} \mid u \text{ adjoins } v\} \\ &= \left\{ \begin{array}{l} \{\text{FL,GA}\},\{\text{FL,AL}\},\{\text{FL,MS}\},\{\text{FL,LA}\},\{\text{GA,AL}\}, \\ \{\text{AL,MS}\},\{\text{MS,LA}\},\{\text{GA,SC}\},\{\text{GA,TN}\},\{\text{SC,NC}\}, \\ \{\text{NC,TN}\},\{\text{MS,TN}\},\{\text{MS,AL}\} \end{array} \right\}. \end{aligned}$$



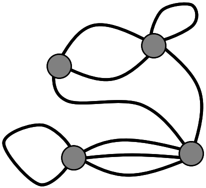
Multigraphs

- Like simple graphs, but there may be more than one edge connecting two given nodes.
- A multigraph $G = (V, E, f)$ consists of a set V of vertices, a set E of edges (as primitive objects), and a function $f : E \to \{\{u, v\} \mid u, v \in V \wedge u \neq v\}\}$.
- E.g., nodes are cities, edges are segments of major highways.



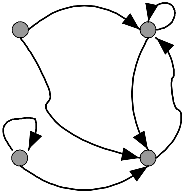
Pseudographs

- Like a multigraph, but edges connecting a node to itself are allowed.
- A pseudograph $G = (V, E, f)$ where $f : E \rightarrow \{\{u, v\} \mid u, v \in V\}$. Edge $e \in E$ is a loop if $f(e) = \{u, u\} = \{u\}$.
- E.g., nodes are campsites in a state park, edges are hiking trails through the woods.



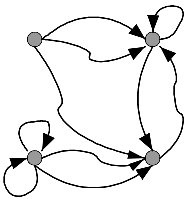
Directed Graphs

- Correspond to arbitrary binary relations R, which need not be symmetric.
- A directed graph (V, E) consists of a set of vertices V and a binary relation E on V.
- E.g.: $V =$ people, $E = \{(x, y) \mid x \text{ loves } y\}$.



Directed Multigraphs

- Like directed graphs, but there may be more than one arc from a node to another.
- A directed multigraph $G = (V, E, f)$ consists of a set V of vertices, a set E of edges, and a function $f : E \to V \times V$.
- E.g., V = web pages, E = hyperlinks. The WWW is a directed multigraph.



Types of Graphs: Summary

- Summary of the book's definitions.
- Keep in mind this terminology is not fully standardized...

Term	Edge type	Multiple edges?	Self-loops?
Simple graph	Undir.	No	No
Multigraph	Undir.	Yes	No
Pseudograph	Undir.	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

Graph Terminology

- Adjacent, connects, endpoints, degree, initial, terminal, in-degree, out-degree, complete, cycles, wheels, *n*-cubes, bipartite, subgraph, union.

Adjacency

- In an undirected graph G, if u and v are two nodes and $e = \{u, v\}$ is an edge in G, then we may say
  - The vertices u and v are *adjacent* (or *neighbors*).
  - The vertices u and v are *endpoints* of the edge e.
  - The edge e is *incident* with the vertices u and v.
  - The edge e *connects* the vertices u and v.

Degree of a Vertex

- Let G be an undirected graph, $v \in V$ a vertex.
- The *degree* of v, denoted by $\deg(v)$, is its number of incident edges. (Except that any self-loops are counted twice.)
- A vertex with degree 0 is *isolated*.
- A vertex of degree 1 is *pendant*.

Handshaking Theorem

- Let G be an undirected (simple, multi-, or pseudo-) graph with vertex set V and edge set E. Then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Corollary Any undirected graph has an even number of vertices of odd degree.

Directed Adjacency

- Let G be a directed (possibly multi-) graph, and $e = (u, v)$ be an edge of G. Then we say:
  - u is *adjacent to* v; v is *adjacent from* u.
  - e *comes from* u; e *goes to* v.
  - e *connects* u *to* v; e *goes from* u *to* v.
  - The *initial vertex* of e is u.
  - The *terminal vertex* of e is v.

Directed Degree

- Let G be a directed graph, and v a vertex of G.
  - The *in-degree* of v, denoted by $\deg^{-}(v)$, is the number of edges going to v.
  - The *out-degree* of v, denoted by $\deg^{+}(v)$, is the number of edges coming from v.
  - The degree of v, $\deg(v) = \deg^{-}(v) + \deg^{+}(v)$, is the sum of v's in-degree and out-degree.

Directed Handshaking Theorem

- Let G be a directed (possibly multi-) graph with vertex set V and edge set E. Then:

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = \frac{1}{2} \sum_{v \in V} \deg(v) = |E|.$$

- Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.

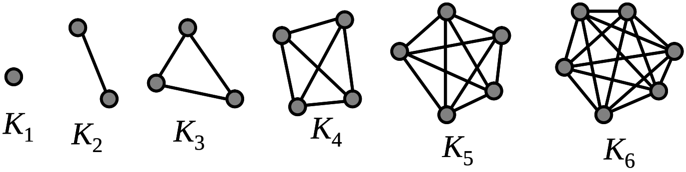
Special Graph Structures

- Special cases of undirected graph structures:
  - Complete graphs K_n.
  - Cycles C_n.
  - Wheels W_n.
  - n-Cubes Q_n.
  - Bipartite graphs.
  - Complete bipartite graphs $K_{m,n}$.

Complete Graphs

- For any $n \in N$, a *complete graph* on n vertices, K_n, is a simple graph with n nodes in which every node is adjacent to every other node:

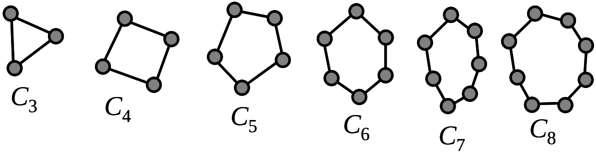
$$\forall u, v \in V : u \neq v \leftrightarrow \{u, v\} \in E.$$



- Note that K_n has $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ edges.

Cycles

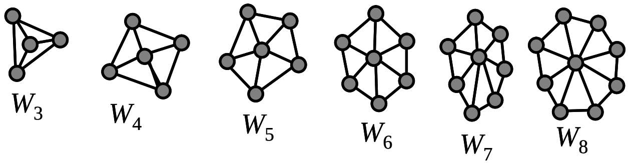
- For any $n \geq 3$, a *cycle* on n vertices, C_n, is a simple graph where $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.



- How many edges are there in C_n?

Wheels

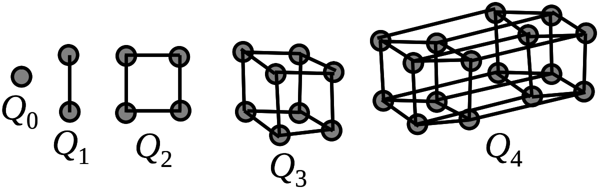
- For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{hub}, v_1\}, \{v_{hub}, v_2\}, \cdots, \{v_{hub}, v_n\}\}$.



- How many edges are there in W_n?

n-Cubes (hypercubes)

- For any $n \in N$, the *hypercube* Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.



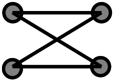
- Number of vertices: 2^n.
- Number of edges: Exercise to try!

n-Cubes (hypercubes) Cont.

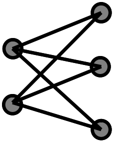
- For any $n \in N$, the hypercube Q_n can be defined recursively as follows:
  - $Q_0 = \{\{v_0\}, \varnothing\}$ (one node and no edges).
  - For any $n \in N$, if $Q_n = (V, E)$, where $V = \{v_1, \cdots, v_a\}$ and $E = \{e_1, \cdots, e_b\}$, then $Q_{n+1} = (V_{Q_{n+1}}, E_{Q_{n+1}})$ is given by
    - $V_{Q_{n+1}} = V \cup \{v'_1, \cdots, v'_a\}$ where $v'_1, \cdots, v'_a$ are new vertices.
    - $E_{Q_{n+1}} = E \cup \{e'_1, \ldots, e'_b\} \cup \{\{v_1, v'_1\}, \{v_2, v'_2\}, \cdots, \{v_a, v'_a\}\}$ where if $e_i = \{v_j, v_k\}$ then $e'_i = \{v'_j, v'_k\}$.
- How many edges are there in an n-cube?

Bipartite Graphs

- A simple graph $G = (V, E)$ is called *bipartite* if V can be partitiovned into two disjoint sets V_1 and V_2 such tht every edge in the graph connects a vertex in V_1 and a vertex in V_2.
- *Complete bipartite graphs*



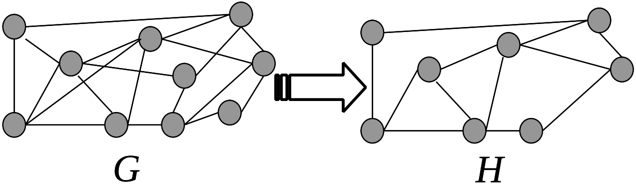
$K_{2,2}$



$K_{2,3}$

Subgraphs

- A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.



Graph Unions

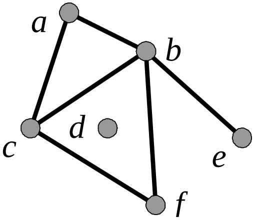
- The union $G_1 \cup G_2$ of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.

What Will Be Given?

- Graph representations
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices.
- Graph isomorphism
  - Two graphs are isomorphic iff they are identical except for their node names.

Adjacency Lists

- *Adjacency Lists*: A table with 1 row per vertex, listing its adjacent vertices.

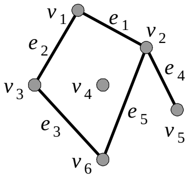


Vertex	Adjacent Vertices
a	b, c
b	a, c, e, f
c	a, b, f
d	
e	b
f	b, c

- *Directed Adjacency Lists*: listing the terminal nodes of each edge incident from that node.

Represented by Matrices

- *Adjacency Matrices*: Matrix $\mathbf{A} = [a_{ij}]$, where a_{ij} is 1 if $\{v_i, v_j\}$ is an edge of G, 0 otherwise.
- *Incidence Matrices*: Matrix $\mathbf{M} = [m_{ij}]$, where m_{ij} is 1 if edge e_i is incident with v_j, 0 otherwise.



$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Graph Isomorphism

Definition

Simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if and only if $\exists$ a bijection $f : V_1 \to V_2$ such that $\forall a, b \in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.

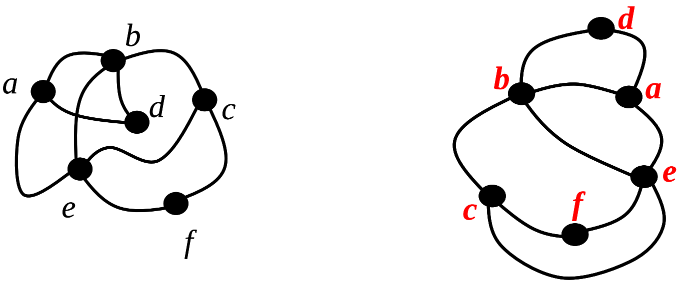
- f is the "renaming" function that makes the two graphs identical.
- The definition can easily be extended to other types of graphs.

Graph Invariants under Isomorphism

- *Necessary* but not *sufficient* conditions for $G_1 = (V_1, E_1)$ to be isomorphic to $G_2 = (V_2, E_2)$.
  - $|V_1| = |V_2|$, $|E_1| = |E_2|$.
  - The number of vertices with degree n is the same in both graphs.
  - For every proper subgraph H of one graph, there is a proper subgraph of the other graph that is isomorphic to H.

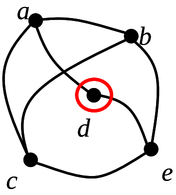
Examples of Isomorphism

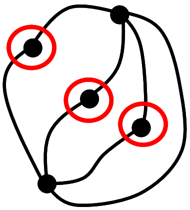
- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.





- Same # of vertices.
- Same # of edges.
- Different # of vertices of degree 2! (1 vs 3)

Paths

- Paths in undirected graphs
  - In an undirected graph, a *path* of length n from u to v is a sequence of adjacent edges going from vertex u to vertex v.
  - A path is a *circuit* if $u = v$.
  - A path *traverses* the vertices along it.
  - A path is *simple* if it contains no edge more than once.
- Paths in directed graphs
  - Same as in undirected graphs, but the path must go in the direction of the arrows.

Connectedness

- An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.
- *Connected component*: connected subgraph
- A cut vertex or cut edge separates 1 connected component into 2 if removed.

Theorem

There is a simple path between any pair of vertices in a connected undirected graph.

Directed Connectedness

- A directed graph is *strongly connected* iff there is a directed path from a to b for any two vertices a and b.
- It is *weakly connected* iff the underlying undirected graph (i.e., with edge directions removed) is connected.
- Note strongly implies weakly but not vice-versa.

Paths & Isomorphism

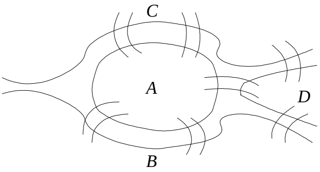
- Note that connectedness, and the existence of a circuit or simple circuit of length k are graph invariants with respect to isomorphism.

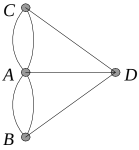
Counting Paths Using Adjacency Matrices

- Let **A** be the adjacency matrix of graph G.
- The number of paths of length k from v_i to v_j is equal to $\left(\mathbf{A}^k\right)_{ij}$. (The notation $(\mathbf{M})_{ij}$ denotes m_{ij} where $[m_{ij}] = \mathbf{M}$.)

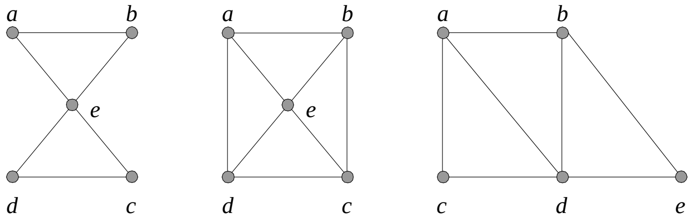
Euler Paths and Circuits

- An *Euler path* in G is a simple path containing every edge of G.
- An *Euler circuit* in a graph G is a simple circuit containing every edge of G.

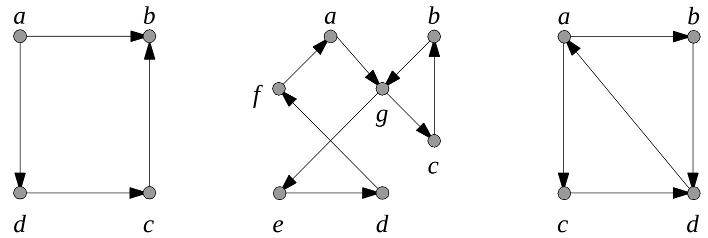




Examples



Examples



Necessary and Sufficient Conditions for Euler Circuits and Paths

Theorem

A connected multigraph has an Euler circuit iff each vertex has even degree.

Theorem

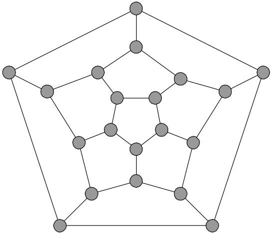
A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.

Constructing Euler Circuits

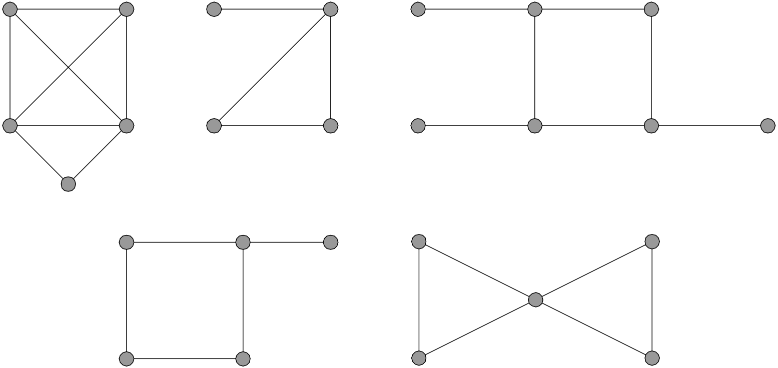
procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G
H := G with the edges of this circuit removed
while H has edges
begin
    subcircuit := a circuit in H
    H := H with the edges of *subcircuit* removed
    circuit := *circuit* with *subcircuit* inserted at the appropriate vertex
end   {*circuit* is an Euler circuit}

Hamilton Paths and Circuits

- A *Hamilton path* is a path that traverses each vertex in G exactly once.
- A *Hamilton circuit* is a circuit that traverses each vertex in G exactly once.



Examples



Some Useful Theorems

Theorem (Dirac's Theorem)

If (but not only if) G is connected, simple, has $n \geq 3$ vertices, and $\forall v : deg(v) \geq n/2$, then G has a Hamilton circuit.

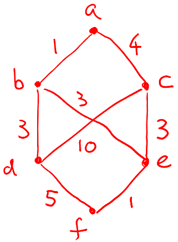
Theorem (Ore's Theorem)

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Shortest-Path Problems

- Weighted graphs $G(V, E, w)$
  - V: a vertex set.
  - E: an edge set.
  - w: a weighting function on E.
- The length of a path, e.g.

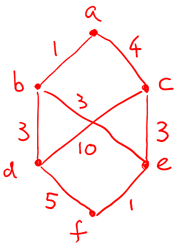
$$\begin{aligned} & w(\{a,b\},\{b,d\},\{d,f\}) \\ = \; & w(\{a,b\}) + w(\{b,d\}) + w(\{d,f\}) \\ = \; & 1+3+5 \\ = \; & 9. \end{aligned}$$



- The shortest path

An Example of the Shortest Path

- Some paths between *a* and *f*.
  - Path ({a,b},{b,d},{d,f}):
    length of ({a,b},{b,d},{d,f}) =
    w({a,b})+w({b,d})+w({d,f}) =
    1+3+5 = 9
  - Path ({a,b},{b,e},{e,f}):
    length of ({a,b},{b,e},{e,f}) =
    w({a,b})+w({b,e})+w({e,f}) =
    1+3+1 = 5
  - Path ({a,c},{c,e},{e,f}):
    length of ({a,c},{c,e},{e,f}) =
    w({a,c})+w({c,e})+w({e,f}) =
    4+3+1 = 8

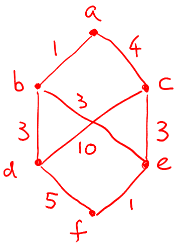


Dijkstra's Algorithm

procedure *Dijkstra*(G: a weighted connected simple graph with all weights positive, a is the source, z is the destination)
// there exists a path from a to z
for $i := 1$ **to** n $L(v_i) := \infty$
$L(a) := 0$
$S := \varnothing$
while $z \notin S$
begin
  $u :=$ a vertex not in S with $L(u)$ minimal
  $S := S \cup \{u\}$
  for all vertices v not in S
    if $L(u) + w(u, v) < L(v)$ then $L(v) := L(u) + w(u, v)$
end $\{L(z) =$ length of a shortest path from a to $z.\}$

Example of Dijkstra's Algorithm

- Find the shortest path between a and f.
- Find the shortest path between d and c.

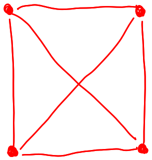


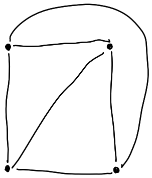
Traveling Salesman Problem

- The traveling salesman problem asks for the circuit of minimum total weight in a weighted, complete, undirected graph that visit each vertex exactly once and returns to its starting point.
- No algorithm with polynomial worst-case time complexity is known.
- c-approximation algorithms: $W \leq W' \leq cW$.
  - W: the total length of an exact solution.
  - W': the total weight of a Hamilton circuit.
  - c: a constant.

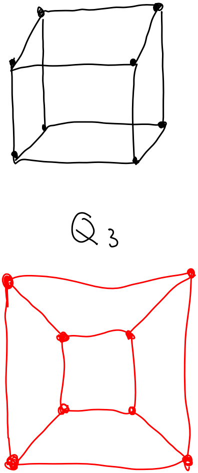
Planar Graphs

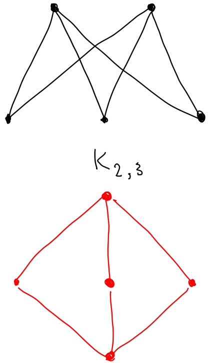
- A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.
- Example:





More Examples





Euler's Formula

- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Corollary

If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Proof.

(1) $2e \geq 3r$. (2) $r = e - v + 2$. □

Euler's Formula (Cont.)

Corollary

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

- Show that K_5 is nonplanar using above corollary.
- Exercise: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Kuratowski's Theorem

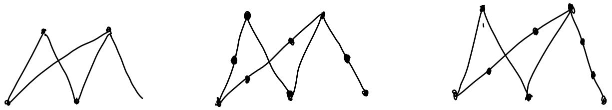
- If a graph is planar, any graph is obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an elementary subdivision.
- The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homeomorphic* if they can be obtained from the same graph by a sequence of elementary subdivision.

Theorem (Kuratowski's Theorem)

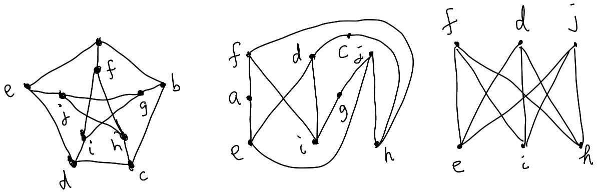
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

Examples

- Some examples

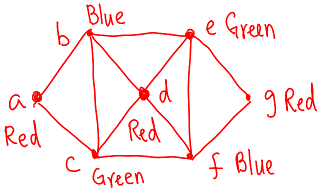


- Is the Petersen graph planar?



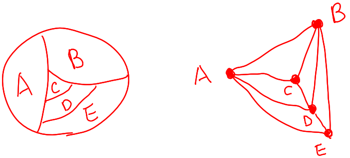
Graph Coloring

- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- The chromatic number of a graph G, denoted by $\chi(G)$, is the least number of colors needed for a coloring of this graph.



Coloring of Maps

- Color a map such that two adjacent regions don't have the same color.
- Each map in the plane can be represented by dual planar graph.
  - Ex,

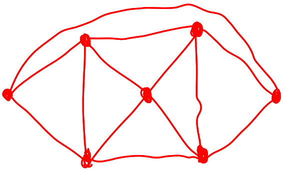


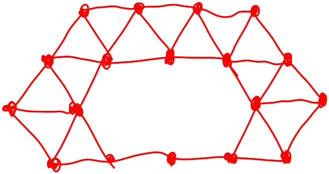
Theorem (The Four Color Theorem)

The chromatic number of a planar graph is no greater than four.

Example

- What is the chromatic number of the graphs?





- What is the chromatic number of K_n?
- What is the chromatic number of $K_{m,n}$?

Frequency Assignment

- Broadcast radio system
  - Two radio stations can't have the same channel if their receiving regions are with some overlapping area.
  - Broadcast stations are represented by vertices.
  - Tow vertices have an edge if their receiving regions are with some overlapping area.
  - Frequency assignment problems is to find the smallest number of channels.