Discrete Mathematics

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§2.3 Functions

On to section 2.3... Functions

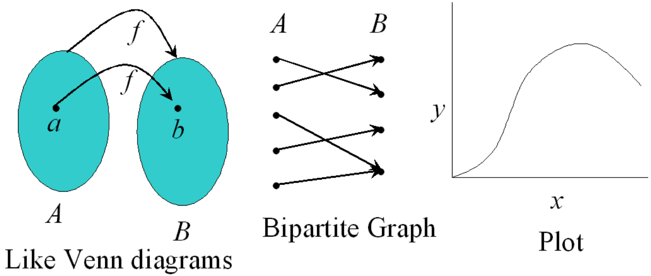
- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x \in R$ a particular value $y = f(x)$, where $y \in R$.
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of any set to elements of any set.

Definition (Functions)

For any sets A, B, we say that a function f from (or "mapping") A to B, denoted as $f : A \longrightarrow B$, is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.

Graphical Representations

- Functions can be represented graphically in several ways:



Functions We've Seen So Far

- A proposition can be viewed as a function from "situations" to truth values $\{\mathbf{T}, \mathbf{F}\}$
  - A logic system called *situation theory*.
  - $p =$"It is raining."; $s =$our situation here, now
  - $p(s) \in \{\mathbf{T}, \mathbf{F}\}$.
- A propositional operator can be viewed as a function from ordered pairs of truth values to truth values. E.g,

$$\vee((\mathbf{F}, \mathbf{T})) = \mathbf{T};$$
$$\rightarrow((\mathbf{T}, \mathbf{F})) = \mathbf{F}.$$

More Functions So Far...

- A *predicate* can be viewed as a function from *objects* to *propositions* (or truth values). E.g.
  - $P :\equiv$ "is 7 feet tall";
  - $P(Mike) =$ "Mike is 7 feet tall." $=$ **F**.
- A *bit string B of length n* can be viewed as a function from the numbers $\{1, \cdots, n\}$ (bit *positions*) to the *bits* $\{0, 1\}$.
  - For $B = 101$, we have $B(3) = 1$.

Still More Functions

- A set S over universe U can be viewed as a function from the elements of U to $\{\mathbf{T}, \mathbf{F}\}$, saying for each element of U whether it is in S. E.g.,
  - $S = \{3\}$; $S(0) = F$, $S(3) = T$.
- "A set operator such as $\cap, \cup, \bar{}$ can be viewed as a function from pairs of sets to sets. E.g.,
  - $\cap(\{1,3\}, \{3,4\}) = \{3\}$.

A Neat Trick

- We denote the set F of *all* possible functions $f : X \to Y$ by Y^X.
  - This notation is especially appropriate, because for finite X, Y, $|F| = |Y|^{|X|}$.
- E.g., if we use representations $\mathbf{F} \equiv 0$, $\mathbf{T} \equiv 1$, $2 :\equiv \{0, 1\} = \{\mathbf{F}, \mathbf{T}\}$, then a subset $T \subseteq S$ is just a function from S to 2, so the power set of S (set of all such fns.) is 2^S in this notation.

Some Function Terminology

- If $f : A \to B$, and $f(a) = b$ (where $a \in A$ and $b \in B$), then:
  - A is the *domain* of f.
  - B is the *codomain* of f.
  - b is the *image* of a under f.
  - a is a *pre-image* of b under f.
    - In general, b may have more than 1 pre-image.
  - The *range* $R \subseteq B$ of f is $\{b \mid \exists a \text{ s.t. } f(a) = b\}$.

Range versus Codomain

- The range of a function might *not* be its whole codomain.
- The codomain is the set that the function is *declared* to map all domain values into.
- The range is the particular set of values in the codomain that the function *actually* maps elements of the domain to.

Range vs. Codomain - Example

- Suppose I declare to you that: "f is a function mapping students in this class to the set of grades $\{A, B, C, D, E\}$."
- At this point, you know f's codomain is: $\{A, B, C, D, E\}$, and its range is unknown!
- Suppose the grades turn out all As and Bs.
- Then the range of f is $\{A, B\}$, but its codomain is still $\{A, B, C, D, E\}$!

Operators (general definition)

- An *n*-ary operator over the set S is any function from the set of ordered *n*-tuples of elements of S, to S itself.
  - E.g., if $S = \{\mathbf{F}, \mathbf{T}\}$, $\neg$ can be seen as a unary operator, and $\wedge, \vee$ are binary operators on S.
  - $\cup$ and $\cap$ are binary operators on the set of all sets.

Constructing Function Operators

- If $\cdot$ ("dot") is any operator over B, then we can extend $\cdot$ to also denote an operator over functions $f : A \rightarrow B$.
  - E.g., Given any binary operator $\cdot : B \times B \rightarrow B$, and functions $f, g : A \rightarrow B$, we define $(f \cdot g) : A \rightarrow B$ to be the function defined by $\forall a \in A$, $(f \cdot g)(a) = f(a) \cdot g(a)$.

Function Operator Example

- $+, \times$ ("plus", "times") are binary operators over R. (Normal addition & multiplication.)
- Therefore, we can also add and multiply functions $f, g : R \to R$:
  - $(f + g) : R \to R$, where $(f + g)(x) = f(x) + g(x)$.
  - $(f \times g) : R \to R$, where $(f \times g)(x) = f(x) \times g(x)$.

Function Composition Operator

- For functions $g : A \rightarrow B$ and $f : B \rightarrow C$, there is a special operator called compose ("$\circ$").
  - It *composes* (creates) a new function out of f, g by applying f to the result of g.
  - $(f \circ g) : A \rightarrow C$, where $(f \circ g)(a) = f(g(a))$.
  - Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$.
  - Note that $\circ$ (like Cartesian $\times$, but unlike $+, \wedge, \cup$) is non-commuting. (Generally, $f \circ g \neq g \circ f$).

Images of Sets under Functions

- Given $f : A \to B$, and $S \subseteq A$,
- The image of S under f is simply the set of all images (under f) of the elements of S. E.g.,

$$
\begin{aligned}
f(S) \quad &:\equiv \{f(s) \mid s \in S\} \\
&:\equiv \{b \mid \exists s \in S : f(s) = b\}.
\end{aligned}
$$

- Note the range of f can be defined as simply the image (under f) of f's domain!

One-to-One Functions

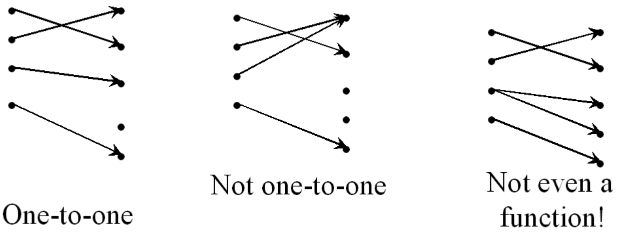
- A function is *one-to-one (1-1)*, or *injective*, or an *injection*, *iff* every element of its range has *only* 1 pre-image.
  - Formally: given $f : A \rightarrow B$,

$$\text{“}x \text{ is injective”} :\equiv (\neg\exists x, y : x \neq y \wedge f(x) = f(y)).$$

- Only *one* element of the domain is mapped to any given *one* element of the range.
- Domain & range have same cardinality. What about codomain?
- Each element of the domain is *injected* into a different element of the range.
  - Compare “each dose of vaccine is injected into a different patient.”

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



Sufficient Conditions for 1-1ness

- For functions f over numbers,
  - f is strictly (or monotonically) increasing iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
  - f is strictly (or monotonically) decreasing iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one. E.g. x^3
  - Converse is not necessarily true. E.g. $1/x$

Onto(Surjective)Functions

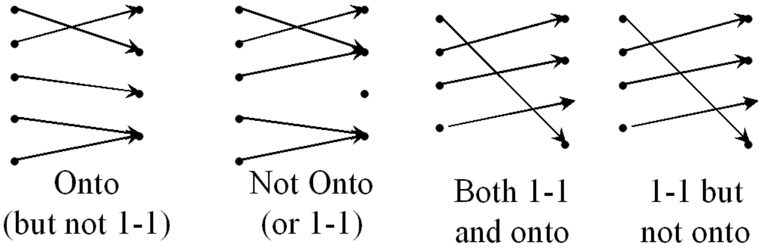
- A function $f : A \rightarrow B$ is *onto* or *surjective* or a *surjection* iff its range is equal to its codomain ($\forall b \in B, \exists a \in A : f(a) = b$).
- An onto function maps the set A *onto* (over, covering) the entirety of the set B, not just over a piece of it.

Example

for domain & codomain R, x^3 is onto, whereas x^2 isn't. (Why not?)

Illustration of Onto

- Some functions that are or are not onto their codomains:



Bijections

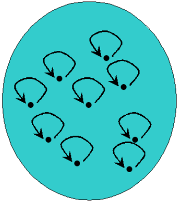
- A function f is a one-to-one *correspondence*, or a *bijection*, or *reversible*, or *invertible*, iff it is both one-to-one and onto.
- For bijections $f : A \longrightarrow B$, there exists an inverse of f, written $f^{-1} : B \longrightarrow A$, which is the unique function such that $f^{-1} \circ f = I$. (the identity function)

The Identity Function

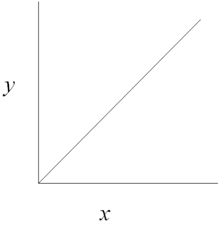
- For any domain A, the identity function $I : A \rightarrow A$ (variously written, $I_A, 1, 1_A$) is the unique function such that $\forall a \in A : I(a) = a$.
- Some identity functions you've seen:
  - $+$ing 0, $\cdot$ing by 1, $\wedge$ing with **T**, $\vee$ing with **F**, $\cup$ing with $\varnothing$, $\cap$ing with U.
- Note that the identity function is both one-to-one and onto (bijective).

Identity Function Illustrations

- The identity function:



Domain and range



Graphs of Functions

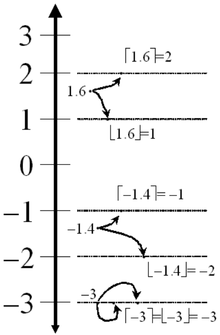
- We can represent a function $f : A \longrightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$.
- Note that $\forall a$, there is only 1 pair $(a, f(a))$.
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane. A function is then drawn as a curve (set of points) with only one y for each x.

A Couple of Key Functions

- In discrete math, we will frequently use the following functions over real numbers:
- $\lfloor x \rfloor$ ("floor of x") is the largest (most positive) integer $\leq x$.
- $\lceil x \rceil$ ("ceiling of x") is the smallest (most negative) integer $\geq x$.

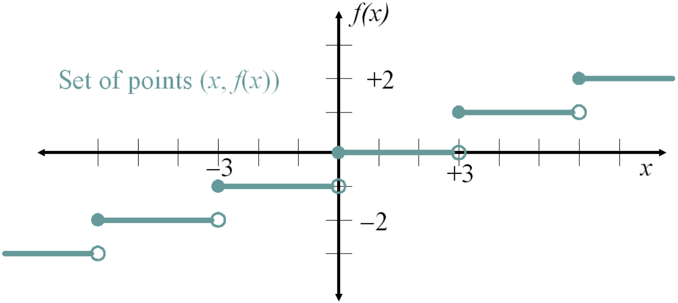
Visualizing Floor & Ceiling

- Real numbers "fall to their floor" or "rise to their ceiling."
- Note that if $x \notin Z$, then $\lfloor -x \rfloor \neq - \lfloor x \rfloor$ & $\lceil -x \rceil \neq - \lceil x \rceil$.
- Note that if $x \in Z$, $\lfloor x \rfloor = \lceil x \rceil = x$.



Plots with floor/ceiling: Example

- Plot of graph of function $f(x) = \lfloor x/3 \rfloor$:



Review of §2.3 (Functions)

- Function variables f, g, h, ...
- Notations: $f : A \to B$, $f(a)$, $f(A)$.
- Terms: image, preimage, domain, codomain, range, one-to-one, onto, strictly (in/de)creasing, bijective, inverse, composition.
- Function unary operator f^{-1} , binary operators $+, -$,etc, and $\circ$.
- The $R \to Z$ functions $\lfloor x \rfloor$ and $\lceil x \rceil$.