Example 1 *Solve the recurrence relation* $a_n = 7a_{n-1} - 10a_{n-2}$ *for* $n \geq 2$ *with* $a_0 = 2$ *and* $a_1 = 1$ *by characteristic equations.*

Solution Let $a_n = r^n$. Then, the C.E. is $r^2 - 7r + 10 = 0$. There are two distinguish root $r = 2$ and $r = 5$. So, the two basic solution is 2^n and 5^n. Assume $a_n = \alpha_1 2^n + \alpha_2 5^n$. From $a_0 = 2$, we have $\alpha_1 + \alpha_2 = 2$. From $a_1 = 1$, we have $2\alpha_1 + 5\alpha_2 = 1$ Thus, we can get $\alpha_1 = 3$ and $\alpha_2 = -1$. So, $a_n = 3 \cdot 2^n - 5^n$.

Example 2 *Solve the recurrence relation* $a_n = 7a_{n-1} - 10a_{n-2} + 16n + 5$ *for* $n \geq 2$ *with* $a_0 = 0$ *and* $a_1 = 4$ *by characteristic equations.*

Solution Let $a_n = h_n + p_n$. From previous results, $h_n = \alpha_1 2^n + \alpha_2 5^n$. Now, assume $p_n = an = b$. Bring p_n into the recurrence relation, then we have

$$(an + b) = 7\left(a\left(n-1\right) + b\right) - 10\left(a\left(n-2\right) + b\right) + 16n + 5.$$

This is a polynomial equation of n. By comparson the coefficent, we can get $a = 4$ and $b = \frac{57}{4}$. Thus,

$$a_n = \alpha_1 2^n + \alpha_2 5^n + 4n + \frac{57}{4}.$$

From $a_0 = 0$, we have $\alpha_1 + \alpha_2 = -\frac{57}{4}$. From $a_1 = 1$, we have $2\alpha_1 + 5\alpha_2 = -\frac{69}{4}$. Thus, we can get $\alpha_1 = -18$ and $\alpha_2 = \frac{15}{4}$. So, $a_n = \left(-18\right) 2^n + \frac{15}{4} 5^n + 4n + \frac{57}{4}$.

Example 3 *Solve the recurrence relation* $a_n = 7a_{n-1} - 10a_{n-2}$ *for* $n \geq 2$ *with* $a_0 = 2$ *and* $a_1 = 1$ *by generating functions.*

Solution

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=2}^{\infty} a_n x^n\right) + a_1 x + a_0 \\
&= \left[\sum_{n=2}^{\infty} \left(7a_{n-1} - 10a_{n-2}\right) x^n\right] + x + 2 \\
&= 7x\left(\sum_{n=2}^{\infty} a_{n-1} x^{n-1}\right) - 10x^2\left(\sum_{n=2}^{\infty} a_{n-2} x^{n-2}\right) + x + 2 \\
&= 7x\left(G(x) - a_0\right) - 10x^2 G(x) + x + 2 \\
&= 7xG(x) - 10x^2 G(x) - 13x + 2
\end{aligned}$$

So, we have $G(x)\left(10x^2 - 7x + 1\right) = -13x + 2$.

$$\begin{aligned}
G(x) &= \frac{-13x + 2}{10x^2 - 7x + 1} = \frac{-13x + 2}{(1-2x)(1-5x)} \\
&= \frac{a}{(1-2x)} + \frac{b}{(1-5x)}.
\end{aligned}$$

Then, we can solve $a = 3$ and $b = -1$.

$$\begin{aligned}
G(x) &= \frac{3}{(1-2x)} - \frac{1}{(1-5x)} = 3\sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} (5x)^n \\
&= \sum_{n=0}^{\infty} \left(3 \cdot 2^n - 5^n\right) 2x^n.
\end{aligned}$$

Thus, $a_n = 3 \cdot 2^n - 5^n$.

Example 4 *Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2^n$ for $n \geq 2$ with $a_0 = 4$ and $a_1 = 12$ by generating function.*

Solution

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=2}^{\infty} a_n x^n\right) + a_1 x + a_0 \\
&= \left[\sum_{n=2}^{\infty} \left(a_{n-1} + 2a_{n-2} + 2^n\right) x^n\right] + 12x + 4 \\
&= x\left(\sum_{n=2}^{\infty} a_{n-1} x^{n-1}\right) + 2x^2\left(\sum_{n=2}^{\infty} a_{n-2} x^{n-2}\right) + \left(\sum_{n=2}^{\infty} (2x)^n\right) + 12x + 4 \\
&= x\left(G(x) - a_0\right) + 2x^2 G(x) + \frac{4x^2}{1-2x} + 12x + 4 \\
&= xG(x) + 2x^2 G(x) + 8x + 4
\end{aligned}$$

So, we have $G(x)\left(-2x^2 - x + 1\right) = \frac{4x^2}{1-2x} - 13x + 2 = \frac{-12x^2+4}{1-2x}$.

$$\begin{aligned}
G(x) &= \frac{-12x^2 + 4}{(1-2x)^2(1+x)} = \frac{a}{(1-2x)^2} + \frac{b}{(1-2x)} + \frac{c}{(1+x)} \\
&= \frac{(-2b+4c)x^2 + (a-b-4c)x + (a+b+c)}{(1-2x)^2(1+x)}.
\end{aligned}$$

Then, we can solve $a = \frac{6}{9}$, $b = \frac{38}{9}$, and $c = -\frac{8}{9}$.

$$\begin{aligned}
G(x) &= \frac{6}{9}\frac{1}{(1-2x)^2} + \frac{38}{9}\frac{1}{(1-2x)} - \frac{8}{9}\frac{1}{(1+x)} \\
&= \left(\sum_{n=0}^{\infty} \frac{6}{9}(n+1)2^n x^n\right) + \left(\sum_{n=0}^{\infty} \frac{38}{9} 2^n x^n\right) - \left(\sum_{n=0}^{\infty} \frac{8}{9}(-1)^n x^n\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{6}{9}(n+1)2^n + \frac{38}{9}2^n + \frac{8}{9}(-1)^n\right) x^n.
\end{aligned}$$

Thus, $a_n = \frac{6}{9}(n+1)2^n + \frac{38}{9}2^n + \frac{8}{9}(-1)^n$.