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Fold principal axis—a new tool for defining the orientations of rotationally symmetric shapes

Sheng-Lin Chou*

Institute of Computer Science and Information Engineering, National Chiao Tung University, Hsinchu, Taiwan 30050, ROC

Ja-Chen Lin and Wen-Hsiang Tsai**

Department of Computer and Information Science, National Chiao Tung University, Hsinchu, Taiwan 30050, ROC

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Abstract

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A new type of principal axis, namely, fold principal axis, is introduced in this paper, which can be used to define the orientations of rotationally symmetric shapes. A given shape is first transformed into a new shape on which the well-defined traditional principal axis is detected. The traditional principal axis is then transformed back to obtain the proposed fold principal axes of the given shape. The properties of such axes, including the uniqueness, existence, and invariance under rotation, translation, and scaling, are investigated. Some illustrative examples are also given.

Keywords. Rotationally symmetric shape, fold-expanded shape, principal axis, fold principal axis, high-order generalized principal axis.

1. Introduction

Rotationally symmetric shapes are frequently encountered in real applications. For example, stars, regular polygons, crosses, gears, etc., are all rotationally symmetric shapes. When a robot is told to pick up an article with a shape of this type in a machine parts assembly work, the robot usually has to locate the article first, including the determination of the position as well as orientation of the article shape. The computation of shape orientations is thus an essential step for many automation applications. For shapes which are not rotationally symmetric, the principal axes of the shapes (Rosenfeld and Kak, 1982) usually give enough information about the orientation of the

* Sheng-Lin Chou is also with Advanced Technology Center, Computer and Communication Research Laboratories, Industrial Technology Research Institute, Chutung, Hsinchu, Taiwan 31015, ROC.
** To whom all correspondence should be sent.

shapes. For shapes which are rotationally symmetric, however, principal axes no more exist (Tsai and Chou, 1990). We thus have a question to answer: can the definition of principal axis be modified so that the orientation of rotationally symmetric shapes can be detected using the modified principal axes? The answer to this question is positive and there exist at least two possible approaches to solving it. The first is to use high-order generalized principal axes, as defined and illustrated in (Tsai and Chou, 1990). Another approach which in general requires less computation time is proposed in this paper. Instead of using high-order generalized principal axes, this new approach still makes use of traditional principal axes and defines a new type of principal axis, called fold principal axis. More specifically, in the proposed approach the traditional principal axis is first computed from a new shape, called fold-expanded shape, transformed from the original shape. The traditional principal axis of the new shape is then detected and transformed back onto the original shape to obtain the proposed fold principal axes which define the orientation of the original shape. Figures will be given to illustrate the idea.

In Section 2 the definitions of rotationally symmetric shapes and fold-expanded shapes will be given. The idea of fold principal axes and several properties of them will also be introduced. In Section 3 the uniqueness of the fold principal axes is proved and the existence of fold principal axes is also investigated. Concluding remarks are given in Section 4.

2. Fold-expanded shape and fold principal axes

A shape S is called an n-fold rotationally symmetric shape (abbreviated as an n-RSS henceforth) if it becomes identical to itself after being rotated around its centroid through any multiple of $2\pi/n$.

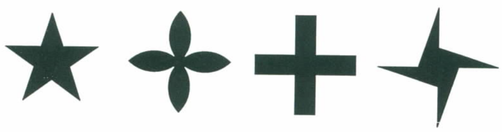

Figure 1. Some common rotationally symmetric shapes.

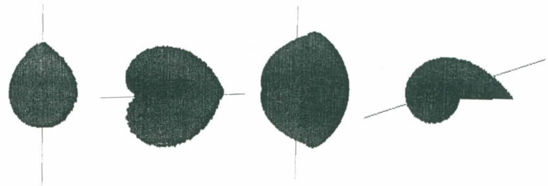

Figure 2. The corresponding fold-expanded shapes of the shapes listed in Figure 1 and their principal axes.

For simplicity, we assume that no other larger integer n has this property. Shapes like those shown in Figure 1 all belong to this class and are frequently encountered in real applications. Throughout this paper, we take the origin of the coordinate system to be the centroid of S. With this convention, the shape S, when sampled in the polar coordinate system, can be described as

$$S = \bigcup_{j=1}^{n} \left\{ (r_i, \theta_{ij}) \mid i = 1, 2, \ldots, m, \text{ and } \theta_{ij} = \theta_{i1} + (j-1)\frac{2\pi}{n} \right\} \quad (1)$$

where m is the number of points in one fold of S and n is the number of mutually disjoint folds contained in S. Here a fold of S is defined as any contiguous area of the n-RSS S bounded by an angle of $2\pi/n$. Therefore, there are totally $n \cdot m$ points in S.

The exact value of n for the given shape S can be detected easily, as is illustrated in (Leou and Tsai, 1987) or (Highnam, 1986). We thus assume that n, the number of folds for S, is a given fixed integer throughout this paper.

It has been proved in (Tsai and Chou, 1990) that the traditional second-order principal axis is undefined for any n-RSS with $n \geqslant 3$. To overcome this difficulty, it is found in this study that we can first transform the original shape S into a new shape E which is not rotationally symmetric (see Figure 2 for examples), then compute the traditional principal axis on E, and finally convert it back into S to define the orientation of S. The details are as follows.

For the n-RSS S described by eq. (1), let $f = f_\varrho$ be a fold of S illustrated by

$$f = f_\varrho = \left\{ (r, \theta) \in S \mid \varrho \leqslant \theta < \varrho + \frac{2\pi}{n} \right\} \quad (2)$$

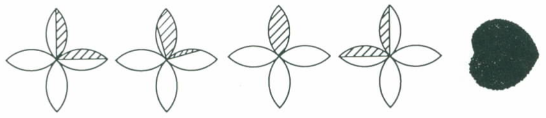

Figure 3. The folds with $\varrho=0$, $\pi/12$, $\pi/4$, and $\pi/2$, respectively (shaded portions) and the identical fold-expanded shape E (the heart shape on the right) expanded by these folds.

where ϱ is any real number lying in the range $[0, 2\pi)$. Then we define the fold-expanded shape generated by f_ϱ to be

$$E_\varrho = \{(r, n\theta) \mid (r,\theta) \in f_\varrho\}. \tag{3}$$

It will be shown in Proposition 1 below that $E_\varrho = E_0$ for all ϱ, i.e., no matter which fold is expanded, the generated fold-expanded shape is always identical, namely, E_0 (see Figure 3 for illustrations). In other words, the fold-expanded shape E of S can be defined as

$$E = E_0. \tag{4}$$

Proposition 1. *$E_\varrho = E_0$ for all ϱ.*

Proof. Let j be the integer satisfying

$$\varrho \in \left[\frac{2\pi}{n} j, \frac{2\pi}{n}(j+1)\right).$$

Let $\varrho_1 \in [0, 2\pi/n)$ be defined as

$$\varrho_1 = \varrho - \frac{2\pi}{n} j.$$

Then

$$\begin{aligned} f_\varrho &= \left\{(r,\theta) \in S \mid \varrho \leqslant \theta < \varrho + \frac{2\pi}{n}\right\} \\ &= \left\{(r,\theta) \in S \mid \varrho \leqslant \theta < (j+1)\frac{2\pi}{n}\right\} \\ &\quad \cup \left\{(r,\theta) \in S \mid (j+1)\frac{2\pi}{n} \leqslant \theta < \varrho + \frac{2\pi}{n}\right\} \\ &= \left\{\left(r_i, \theta_{i1} + j\frac{2\pi}{n}\right) \mid (r_i, \theta_{i1}) \in S \text{ and } \theta_{i1} \in \left[\varrho_1, \frac{2\pi}{n}\right)\right\} \\ &\quad \cup \left\{\left(r_i, \theta_{i1} + (j+1)\frac{2\pi}{n}\right) \mid (r_i, \theta_{i1}) \in S \text{ and } \theta_{i1} \in [0, \varrho_1)\right\} \end{aligned}$$

where θ_{i1} is the one defined in eq. (1). Therefore,

$$\begin{aligned} E_\varrho &= \{(r, n\theta) \mid (r,\theta) \in f_\varrho\} \\ &= \left\{(r_i, n\theta_{i1} + 2\pi j) \mid (r_i, \theta_{i1}) \in S \text{ and } \theta_{i1} \in \left[\varrho_1, \frac{2\pi}{n}\right)\right\} \\ &\quad \cup \{(r_i, n\theta_{i1} + 2\pi(j+1)) \mid (r_i, \theta_{i1}) \in S \text{ and } \theta_{i1} \in [0, \varrho_1)\} \\ &= \left\{(r_i, n\theta_{i1}) \mid (r_i, \theta_{i1}) \in S \text{ and } \theta_{i1} \in \left[0, \frac{2\pi}{n}\right)\right\} \\ &= \{(r, n\theta) \mid (r,\theta) \in f_0\} = E_0. \qquad \square \end{aligned}$$

Proposition 2 below includes two properties of the fold-expanded shape.

Proposition 2. *If an n-RSS S is rotated through an angle of β, then*

(i) *E will be rotated through an angle of $n\beta$, and*

(ii) *the principal axis of E will also be rotated through an angle of $n\beta$.*

Proof. The definition

$$E = \left\{(r, n\theta) \mid (r,\theta) \in S, 0 \leqslant \theta < \frac{2\pi}{n}\right\}$$

implies that if we replace θ by $\theta+\beta$, then the $n\theta$ will be replaced by $n(\theta+\beta) = n\theta + n\beta$ which implies a rotation through an angle of $n\beta$. Statement (ii) is a direct result of (i). □

To define some axes of S in terms of the principal axis of E so that the orientation of S can be defined, one of the basic requirements is that the axes defined should be rotationally invariant. That is, if S is rotated around the centroid of S through an angle of β, then the axes should also be rotated through an angle of β. Based on Proposition 2, we define the fold principal axes of S to be the two half lines through the origin, i.e., through the centroid of S, with direction angles δ_1 and δ_2 specified by

$$\delta_1 = \frac{\phi_1}{n} \tag{5}$$

and

$$\delta_2 = \frac{\phi_2}{n} = \delta_1 + \frac{\pi}{n} \tag{6}$$

(with respect to the X-axis) where ϕ_1 and $\phi_2 = \phi_1 + \pi$ are the two direction angles of the line representing the principal axis of the fold-expanded shape E. If we use the convention that both ϕ_1 and ϕ_2 are in the range $[0, 2\pi)$, then δ_1 and δ_2 will be in the range $[0, 2\pi/n)$.

The traditional principal axis gives two directions (differing from each other by π) for a shape which is not rotationally symmetric; similarly, our definition of the fold principal axes gives two directions (differing from each other by π/n) for a shape which is rotationally symmetric. It is also observed that if we treat a shape which is not rotationally symmetric as a 1-fold shape, i.e., $n=1$, then E is identical to S and the two angles

$$\frac{\phi_1}{n} \quad \text{and} \quad \frac{\phi_2}{n} = \frac{\phi_1}{n} + \frac{\pi}{n}$$

in fact are the direction angles of the traditional principal axis of S. We may thus regard the traditional principal axis as a special case of the fold principal axes with $n=1$.

The property of the invariance to translation and scaling of the fold principal axes is guaranteed by the proposition below.

Proposition 3. *Fold principal axes are invariant under translation and scaling.*

Proof. Assume that a shape S is translated. Since E is constructed using polar coordinates with the pole located at the centroid of S, we know that the position, including the distance and direction, of every point of E is not changed with respect to S. Therefore, the relative position of the principal axis of E, and hence the relative position of the fold principal axes, are not changed with respect to S.

On the other hand, if a shape S is scaled for a factor of λ, then the polar coordinate representations (r, θ) and $(r, n\theta)$ of S and E (see eqs. (1) and (3)) become $(\lambda r, \theta)$ and $(\lambda r, n\theta)$, respectively. Therefore, E is scaled for a factor of λ, too. However, the fact that the traditional principal axis is in general invariant under scaling implies that the two direction angles ϕ_1 and $\phi_1 + \pi$ of the principal axis of E are unchanged. As a result, the fold principal axes of S are invariant by eqs. (5) and (6). □

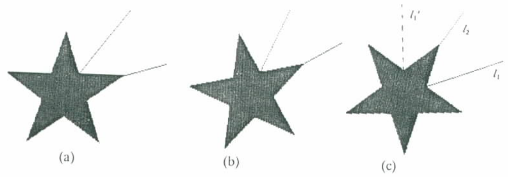

Figure 4. The fold principal axes of a shape with different phases. Notice that the definition of fold-principal axes implies that the direction angles of both axes are in range $[0, 2\pi/n)$. The axis l_1 in (c) is in fact equivalent to the dashed axis l_1' in the sense of modulo $2\pi/n$.

As for the rotational invariance property of the fold principal axes, we have Proposition 4 below.

Proposition 4. *If an n-RSS S is rotated through an angle of β, then the two direction angles*

$$\delta_1 = \frac{\phi_1}{n} \quad \textit{and} \quad \delta_2 = \frac{\phi_2}{n} = \delta_1 + \frac{\pi}{n}$$

of the fold principal axes are replaced by

$$(\delta_1 + \beta)_{\text{mod } 2\pi/n} \quad \textit{and} \quad (\delta_2 + \beta)_{\text{mod } 2\pi/n},$$

respectively.

Proof. If S is rotated through an angle of β, resulting in a new shape S', then Proposition 2 implies that E is rotated through an angle of $n\beta$, resulting in a new fold-expanded shape E'. As the traditional principal axis is rotationally invariant, the difference between the direction angles of the principal axes for E and E' is $n\beta$. That is, the principal axis of E' has directions specified by $\phi_i' =$

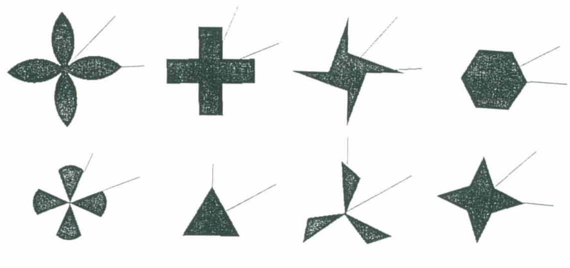

Figure 5. Fold principal axes of several shapes.

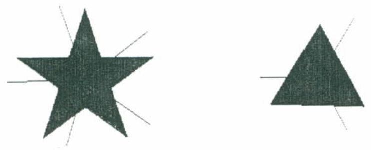

Figure 6. High-order generalized principal axes (see (Tsai and Chou, 1990)) do not necessarily coincide with the two fold principal axes (compare the high-order generalized principal axes in this figure with the fold principal axes of the star shape in Figure 4 and those of the triangle in Figure 5).

$(\phi_i + n\beta)_{\text{mod } 2\pi}$ for both $i=1$ and 2. Therefore, the fold principal axes of S' have directions specified by

$$\delta_i' = \frac{\phi_i'}{n} = \frac{(\phi_i + n\beta)_{\text{mod } 2\pi}}{n} = \left(\frac{\phi_i}{n} + \beta\right)_{\text{mod } 2\pi/n} = (\delta_i + \beta)_{\text{mod } 2\pi/n}$$

for both $i=1$ and 2. □

Notice that the shape S is angularly periodic with period $2\pi/n$ in the sense that the relation

$$\left(r, \theta + \frac{2\pi}{n}\right) \in S$$

holds if and only if a point (r, θ) is in S. Therefore, Proposition 4 has implicitly implied that fold principal axes are invariant under rotation. The idea is illustrated in Figure 4.

In Figure 5, we give the fold principal axes of several rotationally symmetric shapes. Notice that the fold principal axes do not necessarily coincide with any of the n high-order generalized principal axes defined in (Tsai and Chou, 1990). Some examples are given in Figure 6. In that figure, we can see that none of the high-order generalized principal axes agrees with either of the fold principal axes of the star shape shown in Figure 4 or of the triangle shown in Figure 5.

The traditional principal axis is a line, and therefore, has two directions opposite to each other. When such axes are applied to the issue of shape matching, in general a 'double-matching' between an input shape and a reference shape is required, i.e., matching using one of the two direction angles of the principal axes, followed by another matching using the other, should be performed. The same technique may be applied to match rotationally symmetric shapes with the help of the two orientations defined by the fold principal axes.

3. Uniqueness and existence of fold principal axes

The uniqueness of the fold principal axes is assured by Proposition 5 below.

Proposition 5. *Let S be a given n-RSS. Then the two fold principal axes of S are unique.*

Proof. There is a unique E corresponding to the given S. The principal axis of E is a unique line. Let ϕ_1 and $\phi_2 = \phi_1 + \pi$ be the two direction angles of this unique line. Then the two distinct half lines through the centroid of S and with direction angles ϕ_1/n and ϕ_2/n are unique. That is, the two fold principal axes of S are unique by definition. □

As for the existence of the fold principal axes, we first have the following analysis. By the definitions (1) and (4), and also by the definition of fold-expanded shape E, the centroid of E is

$$(\hat{x}, \hat{y}) = \left(\frac{1}{m}\sum_{i=1}^{m} r_i \cos(n\theta_{i1}), \frac{1}{m}\sum_{i=1}^{m} r_i \sin(n\theta_{i1})\right). \quad (7)$$

The definition of the fold principal axes implies that if we want to obtain the direction angles ϕ_1/n and ϕ_2/n of the fold principal axes of S, we must find the direction angles ϕ_1 and $\phi_2 = \phi_1 + \pi$ of the principal axis of E, i.e., we have to (see Rosenfeld and Kak, 1982) find the angle ϕ which minimizes the function I defined by

$$I = \sum_{(x,y)\in E} [(x-\hat{x})\sin\phi - (y-\hat{y})\cos\phi]^2.$$

This I value is minimal when

$$\frac{\partial I}{\partial \phi} = 0, \qquad \frac{\partial^2 I}{\partial \phi^2} > 0$$

are both satisfied, i.e., we have to solve the equation set

$$\tan 2\phi = 2\hat{M}_{11}/(\hat{M}_{20} - \hat{M}_{02}),$$
$$(\hat{M}_{20} - \hat{M}_{02})\cos 2\phi + 2\hat{M}_{11}\sin 2\phi > 0$$

to get ϕ where $\hat{M}_{11}$, $\hat{M}_{20}$ and $\hat{M}_{02}$ are the centralized second-order moments of the fold-expanded shape E. The principal axis of E, and hence, the fold principal axes of S, are well-defined if and only if there exists in the range $[0, 2\pi)$ a unique couple ϕ_1 and $\phi_2 = \phi_1 + \pi$ satisfying this equation set. Therefore, a necessary and sucifficient condition for the existence of the fold principal axes of S is that at least one of the two values $\hat{M}_{11}$ and $\hat{M}_{20} - \hat{M}_{02}$ is not zero. In eq. (1), for every $i = 1, 2, \ldots, m$ let $\hat{x}_i$ and $\hat{y}_i$ be defined as

$$\hat{x}_i = r_i \cos(n\theta_{i1}) \tag{8}$$

and

$$\hat{y}_i = r_i \sin(n\theta_{i1}), \tag{9}$$

respectively. Let $\sum_{i=1}^{m}$ be denoted as $\sum$. By the definitions (1) and (4) again, the centralized moments $\hat{M}_{20}$, $\hat{M}_{11}$, and $\hat{M}_{02}$ are in fact

$$\begin{aligned}\hat{M}_{20} &= \sum (\hat{x}_i - \hat{x})^2 \\ &= (\sum \hat{x}_i^2) - 2\hat{x}(\sum \hat{x}_i) + \sum \hat{x}^2 \\ &= (\sum \hat{x}_i^2) - 2\hat{x}(m\hat{x}) + m\hat{x}^2 \\ &= (\sum \hat{x}_i^2) - m\hat{x}^2 \\ &= (\sum \hat{x}_i^2) - \frac{1}{m}(\sum \hat{x}_i)^2,\end{aligned} \tag{10}$$

$$\begin{aligned}\hat{M}_{11} &= \sum (\hat{x}_i - \hat{x})(\hat{y}_i - \hat{y}) \\ &= (\sum \hat{x}_i\hat{y}_i) - \hat{x}(m\hat{y}) - \hat{y}(m\hat{x}) + \sum \hat{x}\hat{y} \\ &= (\sum \hat{x}_i\hat{y}_i) - m\hat{x}\hat{y} \\ &= (\sum \hat{x}_i\hat{y}_i) - \frac{1}{m}(\sum \hat{x}_i)(\sum \hat{y}_i),\end{aligned} \tag{11}$$

$$\begin{aligned}\hat{M}_{02} &= \sum (\hat{y}_i - \hat{y})^2 \\ &= (\sum \hat{y}_i^2) - 2\hat{y}(\sum \hat{y}_i) + \sum \hat{y}^2 \\ &= (\sum \hat{y}_i^2) - 2\hat{y}(m\hat{y}) + m\hat{y}^2 \\ &= (\sum \hat{y}_i^2) - m\hat{y}^2 \\ &= (\sum \hat{y}_i^2) - \frac{1}{m}(\sum \hat{y}_i)^2.\end{aligned} \tag{12}$$

Therefore, we have the proposition below.

Proposition 6. *A necessary and sufficient condition for the existence of the fold principal axes of a given n-RSS S is that at least one of the following two statements is true*:

$$\text{(i):}\quad \frac{\sum (\hat{x}_i\hat{y}_i)}{m} \neq \frac{\sum \hat{x}_i}{m} \cdot \frac{\sum \hat{y}_i}{m}, \tag{13}$$

i.e., the average of $\{\hat{x}_i\}_{i=1}^{m}$ *times the average of* $\{\hat{y}_i\}_{i=1}^{m}$ *is not equal to the average of* $\{\hat{x}_i\hat{y}_i\}_{i=1}^{m}$, *or equivalently, the average operator and the multiplication operator do not commute,*

$$\text{(ii):}\quad \left(\frac{\sum \hat{x}_i}{m}\right)^2 - \left(\frac{\sum \hat{y}_i}{m}\right)^2 \neq \frac{\sum (\hat{x}_i^2 - \hat{y}_i^2)}{m}, \tag{14}$$

i.e., the square of the average of $\{\hat{x}_i\}_{i=1}^{m}$ *minus the square of the average of* $\{\hat{y}_i\}_{i=1}^{m}$ *is not equal to the average of* $\{\hat{x}_i^2 - \hat{y}_i^2\}_{i=1}^{m}$, *or equivalently, the average operator and the square-difference operator do not commute.*

Proof. We have shown that a necessary and sufficient condition for the existence of the fold principal axes of S is that at least one of the two values $\hat{M}_{11}$ and $\hat{M}_{20} - \hat{M}_{02}$ is not zero. If we divide both sides of eq. (11) by m, we find that the condition $\hat{M}_{11} \neq 0$ occurs if and only if inequality (13) is satisfied. On the other hand, eqs. (10) and (12) imply that $\hat{M}_{20} - \hat{M}_{02} \neq 0$ occurs if and only if the inequality

$$\frac{1}{m}(\sum \hat{x}_i)^2 - \frac{1}{m}(\sum \hat{y}_i)^2 \neq \sum \hat{x}_i^2 - \sum \hat{y}_i^2$$

is satisfied. If we divide both sides by m, we get inequality (14). □

According to our experiments, the fold principal axes of most commonly-seen rotationally symmetric shapes do exist. This means that the proposed fold principal axes are useful for defining the orientations of rotationally symmetric shapes in most applications. In the special case that the fold-expanded shape $E = E_s$ of a given n-RSS S is still a rotationally symmetric shape, the principal axis of E_s will no more be well-defined. This in turn means that S has no fold principal axis. Should this very unusual case happen, the procedure can still be modified in the following way. First perform one more transformation on E_s to obtain the fold-expanded shape of E_s, denoted as E_{E_s}. Next, compute the traditional principal axis of E_{E_s}, and then convert it back onto E_s to obtain the fold principal axes of E_s. Finally obtain the orientation of S by another backward transformation from the direction angles of the fold principal axes of E_s.

4. Concluding remarks

In this paper we have modified the definition of the traditional principal axis and define a new type of axis, called fold principal axis, so that the orientations of rotationally symmetric shapes can be detected using this new type of axis. We first described how to transform the n-fold rotationally symmetric shape S to a new shape E of which the traditional principal axis is well-defined. We then converted the traditional principal axis of E back to S and obtained the fold principal axes of S. Several properties of the fold principal axes have also been studied. A brief discussion about how to apply these axes to the issue of matching rotationally symmetric shapes was given. It is also observed in this study that the uniqueness of the fold principal axes is guaranteed. As for the existence problem of the fold principal axes, a necessary and sufficient condition is provided. The fold principal axes of most common rotationally symmetric shapes were found to be existent. This means that the proposed fold principal axes are useful for applications in which detection of rotationally symmetric shape orientations is necessary. It is noted finally that the computation time required for computing the fold principal axes of an n-RSS S is generally less than that required for computing the nth-order generalized principal axes of S when n is large because only second-order moments are involved in computing the fold principal axes while higher-order moments need be computed to obtain the generalized principal axes. This is an advantage of the proposed approach over (Tsai and Chou, 1990). The detailed analysis to illustrate this fact is too tedious to be included here.

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