DCP 1244 Discrete Mathematics
Lecture 3: Rules of Inference

Tsung Tai Yeh

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Outline

- Mathematical Arguments
- Rules of Inference

Argument

- An **argument** is a sequence of propositions called (**premises**) that end with a conclusion.
- An argument is valid if all its premises are true, then the conclusion is true.
- Example, "If it is Sunday, I don't want to go to school".
  - "$\because$ It is Sunday."
  - "$\therefore$ I don't want to go to school."

Valid Argument Form

- We can express the previous example as following form.
  - Let p: It is Sunday, q: I don't want to go to school.

$$
\begin{array}{l}
p \longrightarrow q \\
p \\
\hline
\therefore\ q
\end{array}
$$

- The above form is valid no matter what propositions are substituted to the variables
  - Both $p \rightarrow q$ and p are true, then q must also be true.
  - The conclusion is true if the premises are all true.
  - We call this form as valid argument form.

Valid Argument Form

- By the definition, the argument form consists of
  - Premises: $p_1, p_2, ..., p_n$
  - Conclusion: q
  - $(p_1 \wedge p_2 \wedge ... \wedge p_n) \rightarrow q$ is a tautology
  - Tautology is a formula which truth values are always true.
  - Example: $((p \rightarrow q) \wedge p) \rightarrow q$ is tautology.

Rules of Inference

- It is impractical to only use the truth table to if an argument form is valid
  - An argument form with 10 different propositional variables requires 2^{10} rows.
- **Rules of inference**
  - Can be used to construct more complicated valid argument form.
  - How to determine $(p \wedge (p \to q)) \to q$ is tautology ?
  - Using **modus ponens** with following valid argument form.

$$\begin{array}{rl} & p \\ & p \longrightarrow q \\ \hline \therefore & q \end{array}$$

Rules of Inference – Modus ponens

- **Modus ponens (method of affirming)**
  - **premises**: p, $p \to q$
  - **conclusion**: q
  - **Tautology**: $(p \land (p \to q) \to q)$
  - Let p: It is snowing today, $p \to q$: If it snows today, then we will go skiing.
  - p is true, by modus poens, $p \to q$ is true. Therefore, q is true.

Rules of Inference

- **Modus tollens (method of denying)**
  - **premises**: $\neg q$, $p \to q$
  - **conclusion**: $\neg p$
  - **Tautology**: $(\neg q \wedge (p \to q)) \to \neg p$
- **Hypothetical Syllogism**
  - **premises**: $p \to q$, $q \to r$
  - **conclusion**: $p \to r$
  - **Tautology**: $((p \to q) \wedge (q \to r)) \to (p \to r)$
- **Disjunctive Syllogism**
  - **premises**: $p \vee q$, $\neg p$
  - **conclusion**: q
  - **Tautology**:$((p \vee q) \wedge \neg p) \to q$

Rules of Inference

- **Addition**
  - **premises**: p
  - **conclusion**: $p \vee q$
  - **Tautology**:$p \rightarrow (p \vee q)$
- **Simplification**
  - **premises**: $p \wedge q$
  - **conclusion**: p
  - **Tautology**:$(p \wedge q) \rightarrow p$

Rules of Inference

- **Conjunction**
  - **premises**: p, q
  - **conclusion**: $p \wedge q$
  - **Tautology**:$((p) \wedge (p)) \rightarrow (p \wedge q)$
- **Resolution**
  - **premises**: $p \vee q$, $\neg p \vee r$
  - **conclusion**: $q \vee r$
  - **Tautology**:$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

Test Yourself

- State which rule of inference is used in the argument?

  - If it rains today, then we will not have a BBQ today. If we do not have a BBQ today, then we will have a BBQ tomorrow. Therefore, if it rains today, then we will have a BBQ tomorrow.

Applying Rules of Inference

- **Example 1**: It is known that

  - It is not sunny this afternoon and it is colder than yesterday.
  - We will go swimming only if it is sunny.
  - If we do not go swimming, then we will take a canoe trip.
  - If we take a canoe trip, then we will be home by sunset.
- Could we conclude that "We will be home by sunset" ?

Solution

- Let's simplify our discussion
  - p:= It is sunny this afternoon.
  - q:= It is colder than yesterday.
  - r:= We will go swimming.
  - s:= We will take a canoe trip.
  - t:= We will be home by sunset.
- A valid argument is given by:
  - **premises**: $\neg p \wedge q$, $r \to p$, $\neg r \to s$, $s \to t$
  - **conclusion**: t

Solution

- Solving step by step:

	Step	Reason
1.	$\neg p \land q$	Premise
2.	$\neg p$	Simplification using (1)
3.	$r \to p$	Premise
4.	$\neg r$	Modus tollens using (2)(3)
5.	$\neg r \to s$	Premise
6.	s	Modus ponens using (4)(5)
7.	$s \to t$	Premise
8.	t	Modus ponens using (6)(7)

- We can conclude t: "We will be home by sunset" is true by using rules of inference.

Applying Rules of Inference

- **Example 2**: It is known that

  - If you send me an e-mail message, then I will finish writing the program.

  - If you do not send me an e-mail message, then I will go to sleep early.

  - If I go to sleep early, then I will wake up feeling refreshed.

- Could we conclude that "If I do not finish writing the program, then I will wake up feeling refreshed ?"

Solution

- To simplify the discussion, let
  - p:= You send me an e-mail message.
  - q:= I will finish writing the program.
  - r:= I will go to sleep early.
  - s:= I will wake up feeling refreshed.
- A valid argument is given by:
  - **premises**: $p \to q$, $\neg p \to r$, and $r \to s$
  - **conclusion**: $\neg q \to s$

Solution

- Solving step by step:

	Step	Reason
1.	$p \rightarrow q$	Premise
2.	$\neg q \rightarrow \neg p$	Contrapositive of (1)
3.	$\neg p \rightarrow r$	Premise
4.	$\neg q \rightarrow r$	Hypothetical syllogism (2)(3)
5.	$r \rightarrow s$	Premise
6.	$\neg q \rightarrow s$	Hypothetical syllogism (4)(5)

- This argument form shows that the premises lead to the desired conclusion.

Rules of Inference with Quantifiers

- **Universal instantiation**
  - **premises**: $\forall x P(x)$
  - **conclusion**: $P(c)$ for any c
  - The rule of inference that is used to conclude that $P(c)$ is true, where c is a particular member of the domain.
  - Example: To conclude from the statement "All women are wise" that "Lisa is wise".
  - Lisa is a member of the domain of all women.
- **Universal generalization**
  - **premises**: $P(c)$ for any **arbitrary** c
  - $\forall x P(x)$

Rules of Inference with Quantifiers

- **Existential instantiation**
  - **premises**: $\exists x P(x)$
  - **conclusion**: P(c) for some element c
  - Let us conclude there is an element c in the domain for which P(c) is true if we know that $\exists x P(x)$ is true.
- **Existential generalization**
  - **premises**: P(c) for some element c
  - **conclusion**: $\exists x P(x)$
  - When we know one element c in the domain where $P(c)$ is true, then we know that $\exists x P(x)$ is true.

Applying Rules of Inferences

- Example 3: It is known that
  - A student in this class has not read the book.
  - Everyone in this class passed the first exam.
- Can we conclude "Someone who passed the first exam has not read the book" ?

Solution

- To simplify the discussion, let
  - $C(x)$: x is in this class.
  - $B(x)$: x has read the book.
  - $P(x)$: x passed the first exam.
- We will give a valid argument with
  - **premises**: $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$
  - **conclusion**: $\exists x(P(x) \wedge \neg B(x))$

Solution

- Solving step by step:

	Step	Reason
1.	$\exists x(C(x) \land \neg B(x))$	Premise
2.	$C(a) \land \neg B(a)$	Existential instantiatiation (1)
3.	$C(a)$	Simplification (2)
4.	$\forall x(C(x) \to P(x))$	Premise
5.	$C(a) \to P(a)$	Universal instantiation (4)
6.	$P(a)$	Modus ponens (3)(5)
7.	$\neg B(a)$	Simplification (2)
8.	$P(a) \land \neg B(a)$	Conjunction (6)(7)
9.	$\exists x(P(x) \land \neg B(x))$	Existential generalization (8)