DCP 1244 Discrete Mathematics
Lecture 13: Generating Function

Tsung Tai Yeh

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Outline

- Generating Function
- Counting Problems
- Solving Recurrence Relations

What is Generating Functions ?

- Generating functions are used to represent sequences efficiently
  - Coding sequences as the formal series
  - Solving many types of counting problems
  - Translating recurrence relation into a sequence
- **Ordinary Generating Function**
  - Given the sequence $a_0, a_1, ..., a_k, ... \in R$ is the infinite series.
  - $G(x) = a_0 + a_1 x + ... + a_k x^k + ... = \sum_{k=0}^{\infty} a_k x^k$

Case Study

- What is the generating function for the sequence 1, 1, 1, 1, 1, 1 ?
  - The generating function of this sequence is $1 + x + x^2 + x^3 + x^4 + x^5$
  - $G(x) = (x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$
- Let m be a positive integer. Let $a_k = C(m, k)$, where $k = 0, 1, 2, ..., m$. What is the generating function for the sequence $a_0, a_1, ..., a_m$?
  - $G(x) = C(m, 0) + C(m, 1)x, C(m, 2)x^2 + ... + C(m, m)x^m$
  - By **Binomial theorem**, $G(x) = (1 + x)^m$

Case Study

- $f(x) = 1/(1-x)$ is the generating function of the sequence 1, 1, 1, ..., why?
  - $\sum_{x=0}^{\infty} x^n = 1/(1-x) = 1 + x + x^2 + ...$
  - $|x| < 1$, by Taylor's Series
- $f(x) = 1/(1-ax)$ is the generating function of the sequence $1, a, a^2, a^3, ...$ why?
  - $\sum_{k=0}^{\infty} a_k x^k = 1/(1-ax) = 1 + ax + a^2x^2 + ...$
  - $|x| < 1/|a|, a \neq 0$

Operations of Generating Functions

- **Theorem:** Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$
  - $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$
  - $f(x) g(x) = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_j b_{k-j}) x^k$
- **Example:** Let $f(x) = 1/(1-x)^2$, find $a_0, a_1, ...$ in the expansion of $f(x) = \sum_{k=0}^{\infty} a_k x^k$
  - We have known $1/(1-x) = 1 + x + x^2 + ...$
  - $1/(1-x)^2 = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} 1) x^k = \sum_{k=0}^{\infty} (k+1) x^k$

Extended Binomial Coefficient

- **The extended binomial coefficient** $\binom{u}{k}$, where $u \in \mathrm{R}, k \in \mathrm{Z}^+$, is defined by
  - $\binom{u}{k} = \begin{cases} u(u-1)...(u-k+1)/k!, & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$
  - $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$
- Find the values of the extended binomial coefficients $\binom{-2}{3}$
  - $\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$

The Extended Binomial Theorem

- **Theorem (The extended binomial theorem):** Let $x, u \in \mathrm{R}, |x| < 1$
  - $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$
- **Example:** Find the generating functions for $(1+x)^{-n}$ and $(1-x)^{-n}$, where $n \in \mathrm{Z}^+$.
  - $(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k$
  - $(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k C(n+k-1, k) x^k$
  - $(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1, k) x^k$

Counting Problems

- Using generating functions to find **the number of k-combinations of a set with n elements**.

  - The generating function: $f(x) = \sum_{k=0}^{n} a_k x^k = (1+x)^n$

  - a_k represents the number of k-combinations of a set with n elements

  - By the **binomial theorem:** $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

  - **C(n, k)** is the number of k-combinations of a set with n elements.

Counting Problems

- Using generation functions to find **the number of r-combinations from a set with n elements when repetition of elements is allowed**.
  - $G(x) = \sum_{r=0}^{\infty} a_r x^r = (1 + x + x^2 + \ldots)^n$
  - We have known $1 + x + x^2 + \ldots = 1/(1-x)$, where $|x| < 1$
  - $G(x) = 1/(1-x)^n = (1-x)^{-n}$
  - By **the extended binomial theorem:**
  - $(1-x)^{-n} = (1 + (-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r$
  - $\binom{-n}{r} (-1)^r = (-1)^r C(n+r-1, r) \cdot (-1)^r = C(n+r-1, r)$

Counting Problems

- Using generating functions to find **the number of ways to select r objects of n different kinds if we must select at least one object of each kind**.
  - $$G(x) = (x+x^2+x^3+\ldots)^n = x^n(1+x+x^2+\ldots)^n = x^n/(1-x)^n$$
  - $G(x) = x^n/(1-x)^n = x^n \cdot (1-x)^{-n}$
  - $= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r$
  - $= \sum_{r=0}^{\infty} C(n+r-1, r) x^{n+r}$
  - $= \sum_{t=n}^{\infty} C(t-1, t-n) x^t$
  - $= \sum_{r=n}^{\infty} C(r-1, r-n) x^r$

Solving Recurrence Relations

- Solving the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, ...,$ and the initial condition $a_0 = 2$
  - $G(x) = \sum_{k=0}^{\infty} a_k x^k$ be the generating function of $\{a_k\}$
  - $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$
  - $G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3 \sum_{k=1}^{\infty} a_{k-1} x^k =$

  $a_0 + \sum_{k=1} (a_k - 3a_{k-1}) x^k = 2$
  - $G(x) - 3xG(x) = (1 - 3x)G(x) = 2$
  - $G(x) = 2/(1 - 3x)$ and $1/(1 - ax) = \sum_{k=0}^{\infty} a^k x^k$
  - $G(x) = 2 \sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k$
  - $a_k = 2 \cdot 3^k$

Solving Recurrence Relations

- Solving $a_n = 8a_{n-1} + 10^{n-1}$, where $a_1 = 9, a_0 = 1$
  - $a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n$
  - $G(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ be the generating function of a_n
  - $G(x) - 1 = \sum\limits_{n=1}^{\infty} a_n x^n$
  - $= \sum\limits_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n)$
  - $= 8x \sum\limits_{n=1}^{\infty} a_{n-1}x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$
  - $= 8x \sum\limits_{n=0}^{\infty} a_n x^n + x \sum\limits_{n=0}^{\infty} 10^n x^n$
  - $= 8xG(x) + x/(1 - 10x)$

Solving Recurrence Relations

- Solving $a_n = 8a_{n-1} + 10^{n-1}$, where $a_1 = 9, a_0 = 1$
  - $G(x) - 1 = 8xG(x) + x/(1-10x)$
  - $G(x) = \frac{1-9x}{(1-8x)(1-10x)}$
  - $G(x) = \frac{1}{2}(\frac{1}{1-8x} + \frac{1}{1-10x})$
  - $G(x) = \frac{1}{2}(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$
  - $a_n = \frac{1}{2}(8^n + 10^n)$