DCP 1244 Discrete Mathematics
Lecture 12: Recurrence

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Outline

- Sequences
- Recurrence Relations
- Linear Recurrence Relations

What is a Sequence ?

- A discrete structure used to represent an ordered list.
  - e.g. a finite sequence: 1, 2, 3, 5. an infinite sequence: 1, 3, 9, ..., 3^n...
  - Using the notation $\{a_n\}$ to describe the sequence.
- **Geometric progression**
  - $a, ar, ar^2, ..., ar^n, ...$
  - The initial term is a, and the common ratio is r, $a, r \in$ R
- **Arithmetic progression**
  - $a, a + d, a + 2d, ..., a + nd, ...$
  - The initial term is a, and the common difference is d

Recurrence Relations

- A recurrence relation is an equation that express a_n in terms of one or more of the previous terms of the sequence, $a_0, a_1, ..., a_{n-1}$, for integer n with $n \geq n_0$, where $n_0 \in Z^+$.
- **initial condition** specifies the terms that precede the first term where the recurrence relation takes effect.
- **Example:** $a_n = a_{n-1} + 3$, where $n = 1, 2, 3, ...$ and $a_0 = 2$ (a_0: initial condition)
  - $a_1 = a_0 + 3 = 5,\ a_2 = 5 + 3 = 8,\ a_3 = 8 + 3 = 11$
- **Example:** $a_n = a_{n-1} - a_{n-2}$**, where** $n = 2, 3, 4, ...$**,** $a_0 = 3, a_1 = 5$ (a_0, a_1: initial condition)
  - $a_2 = a_1 - a_0 = 5 - 3 = 2, a_3 = a_2 - a_1 = 2 - 5 = -3$

Fibonacci Sequence

- The **Fibonacci sequence** is defined by the initial conditions $f_0 = 0,\ f_1 = 1$
  - The recurrence relation: $f_n = f_{n-1} + f_{n-2}$, where $n \geq 2$
  - $f_2 = f_1 + f_0 = 1 + 0 = 1$
  - $f_3 = f_2 + f_1 = 1 + 1 = 2$
  - $f_4 = f_3 + f_2 = 2 + 1 = 3$
  - $f_5 = f_4 + f_3 = 3 + 2 = 5$
  - $f_6 = f_5 + f_4 = 5 + 3 = 8$

Closed Formula

- The closed formula is used to solve the recurrence relation with the initial conditions for the terms of the sequence.
- What is the closed formula of $a_n = a_{n-1} + 3$, where $n \geq 1$?
  - Initial condition $a_0 = 2$
  - **forward substitution**
  - $a_2 = 2 + 3$
  - $a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$
  - $a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$
  - $a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) = 2 + 3(n - 1)$

Closed Formula

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- What is the closed formula of $a_n = a_{n-1} + 3$, where $n \geq 1$?
  - Initial condition $a_0 = 2$
  - **backward substitution**
  - $a_n = a_{n-1} + 3$
  - $(a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$
  - $(a_{n-3}) + 3 + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$
  - $a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)$

Test Yourself

- A person deposits \$10,000 at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years ?
  - Let P_n denote the amount in the account after n years.
  - $P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}$
  - $P_1 = (1.11)P_0$
  - $P_2 = (1.11)P_1 = (1.11)^2 P_0$
  - $P_3 = (1.11)P_2 = (1.11)^3 P_0$
  - $P_n = (1.11)P_{n-1} = (1.11)^n P_0$
  - $P_{30} = (1.11)^{30} \cdot 10,000$

Linear Recurrence Relations

- A linear homogeneous recurrence relation of **degree k** with constant coefficients is a recurrence relation of the form:
  - $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$
  - $c_1, c_2, ..., c_k \in R, c_k \neq 0$
- The **Linear** recurrence relation
  - The right hand side is a sum of previous terms of the sequence each multiplied by a function of n.
- The **homogeneous** recurrence relation
  - No terms occur that are not multiples of the $a_j s$.
- The coefficients are all **constant** in terms of the sequence rather than functions that depend on n.
- The **degree** of the recurrence relation
  - a_n is expressed in terms of the previous k terms of the sequence.

Linear Recurrence Relations

- What is the degree of the following recurrence relation?
  - $P_n = (1.11)P_{n-1}$, P_n degree: 1
  - $f_n = f_{n-1} + f_{n-2}$, f_n degree: 2
  - $a_n = a_{n-5}$, a_n degree: 5
- What is linear recurrence relations?
  - $a_n = a_{n-1} + a_{n-2}^2$ is not linear.
  - $H_n = 2H_{n-1} + 1$ is not homogeneous.
  - $B_n = nB_{n-1}$ does not have constant coefficient.

Characteristic equation of Recurrence Relation

- The recurrence relations have solutions of the form $a_n = r^n$, where r is a constant.

  - $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if
  - $r^n = c_a r^{n-1} + c_2 r^{n-2} + ... + c_k r^{n-k}$.
  - $r^k - c_1 r^{k-1} - c_2 r^{k-2} - .. - c_{k-1} r - c_k = 0$ when both side of the above equation divided by $r^{n-k}, r \neq 0$.

- **Characteristic equation**

  - The sequence $\{a_n\}$ with $a_n = r^n$, where $r \neq 0$ is a solution iff r is a solution of $r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_{k-1} r - c_k = 0$.

- **Characteristic roots**

  - The solution of the characteristic equation.

The Degree Two Case

- **A solution of the recurrence relation**
  - $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
  - $n = 0, 1, 2, \ldots, \alpha_1, \alpha_2$ are constants.
  - $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2, where $c_1, c_2 \in R$.
- **Proof:**

$$
\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} \quad &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r^{n-2} + \alpha_2 r_2^{n-2}) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n
\end{aligned}
$$

The Degree Two Case

- There are constants α_1 and α_2 such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
  - **initial condition:**
  - $a_0 = C_0 = \alpha_1 + \alpha_2$.
  - $a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$.
- Solving these two equations for α_1 and α_2
  - $\alpha_2 = C_0 - \alpha_1$
  - $C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2 = \alpha_1 (r_1 - r_2) + C_0 r_2$
  - $\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$
  - $\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2}$

Case Study

- What is the solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2} = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
  - The characteristic equation: $r^2 - c_1 r - c_2 = r^2 - r - 2 = 0$
  - $r = 2, r = -1$
  - $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$
  - $\alpha_0 = 2 = \alpha_1 + \alpha_2$
  - $\alpha_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$
  - $\alpha_1 = 3, \alpha_2 = -1$
  - $a_n = 3 \cdot 2^n - (-1)^n$

Case Study

- The recurrence relation of Fibonacci numbers: $f_n = f_{n-1} + f_{n-2}$
  - **Initial condition:** $f_0 = 0, f_1 = 1$
  - **Characteristic equation:** $r^2 - r - 1 = 0$
  - $r_1 = \frac{(1+\sqrt{5})}{2}, r_2 = \frac{(1-\sqrt{5})}{2}$
  - $f_n = \alpha_1(\frac{(1+\sqrt{5})}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$
  - $f_0 = \alpha_1 + \alpha_2 = 0$
  - $f_1 = \alpha_1(\frac{1+\sqrt{5}}{2} + \alpha_2(\frac{1-\sqrt{5}}{2})) = 1$
  - $\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = -\frac{1}{\sqrt{5}}$
- Fibonacci numbers are given by:
  - $f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

One Characteristic Root of Multiplicity Two

- **Theorem:** $r^2 - c_1 r - c_2 = 0$ has only one root r_0, where $c_1, c_2 \in \mathrm{R}, c_2 \neq 0$
  - A solution of recurrence relation:
  - $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff
  - $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$
  - $n = 0, 1, 2, ...,$ α_1, α_2 are constants.

Case Study

- What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial condition $a_0 = 1, a_1 = 6$?
  - **Characteristic equation:** $r^2 - 6r + 9 = 0$
  - The only root $r = 3$
  - The solution: $a_n = \alpha_1 3^n + \alpha_2 n 3^n$.
  - $a_0 = 1 = \alpha_1$
  - $a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$
  - $\alpha_1 = 1, \alpha_2 = 1$
- The solution of this recurrence relation a_n
  - $a_n = 3^n + n3^n$

The General Case

- **Theorem:** Let $c_1, c_2, ..., c_k \in R$
  - **Characteristic equation:** $r^k - c_1 r^{k-1} - ... - c_k = 0$
  - k distinct roots $r_1, r_2, ..., r_k$
  - $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if
  - $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + ... + \alpha_k r_k^n$
  - $n = 0, 1, 2, ...,$ where $\alpha_1, \alpha_2, ..., \alpha_k$ are constants.

Case Study

- What is the solution of this recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with initial condition $a_0 = 2, a_1 = 5, a_2 = 15$?
  - **Characteristic polynomial:** $r^3 - 6r^2 + 11r - 6$
  - The characteristic roots are $r = 1, r = 2, r = 3$
  - $a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n$
  - $a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$
  - $a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3$
  - $a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9$
  - $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2$
  - $a_n = 1 - 2^n + 2 \cdot 3^n$

Case with m Multiplicity of the Root

- **Theorem:** Let $c_1, c_2, ..., c_k \in$ R and the characteristic equation
  - $r^k - c_1 r^{k-1} - ... - c_k = 0$
  - t distinct roots $r_1, r_2, ..., r_t$ with multiplicities $m_1, m_2, ..., m_t$
  - $m_i \geq 1, i = 1, 2, ..., t, t,\ m_1 + m_2 + ... + m_t = k$
  - $a_n = c_1 a_{n-1} + c2 a_{n-2} + ... + c_k a_{n-k}$ if and only if
  - $a_n = (\alpha_{1,0} + \alpha_{1,1} n + ... + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + ... + \alpha_{2,m_2-1}) r_2^n + ... + (\alpha_{t,0} + \alpha_{t,a} n + ... + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$
  - $n = 0, 1, 2, ...,$ where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$

Case Study

- The roots of the characteristic equation in a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, 9 (the root 2, 5, 9 with the multiplicity 3, 2, 1, respectively.) What is the form of the general solution ?
  - The general form of the solution:
  - $(\alpha_{1,0} + \alpha_{1.1} n + \alpha_{1,2} n^2)2^n + (\alpha_{2,0} + \alpha_{2,1} n)5^n + \alpha_{3,0} 9^n$

Case Study

- What is the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1, a_1 = -2, a_2 = -1$?
  - **Characteristic equation:** $r^3 + 3r^2 + 3r + 1 = (r+1)^3$
  - $r = -1$ with multiplicity 3
  - $a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n$
  - $a_0 = 1 = \alpha_{1,0}$
  - $a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$
  - $a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$
  - $\alpha_{1,0} = 1, \alpha_{1,1} = 3, \alpha_{1,2} = -2$
  - $a_n = (1 + 3n - 2n^2)(-1)^n$

Linear Nonhomogeneous Recurrence Relation

- **Linear nonhomogeneous recurrence relation with constant coefficients**
  - $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$
  - $F(n)$ is a function not identically zero depending only on n.
  - e.g. $a_n = 3a_{n-1} + 2n$
- **Associated homogeneous recurrence relation**
  - $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$
  - e.g. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Linear Nonhomogeneous Recurrence Relation

- $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients
- Every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$
- $\{a_n^h\}$ is a solution of the associated homogeneous recurrence relation
- **Proof:**
  - $a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + ... + c_k a_{n-k}^{(p)} + F(n)$
  - $b_n = c_1 b_{n-1} + c_2 b_{n-2} + ... + c_k b_{n-k} + F(n)$
  - $b_n - a_n^{(p)} =$
  $c(b_{n-1}) - a_{n-1}^{(p)} + c_2(b_{n-2} - a_{n-2}^{(p)} + ... + c_k(b_{n-k}) - a_{n-k}^{(p)})$
  - $\{b_n - a_n^{(p)}\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$
  - $b_n = a_n^{(p)} + a_n^{(h)}$

Case Study

- Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.

  - The associated linear homogeneous equation is

  $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

  - $F(n) = 2n$ is a polynomial in n of degree one.
  - Supposed that $p_n = cn + d$ is a solution.
  - $a_n = 3a_{n-1} + 2n =¿\ cn + d = 3(c(n-1) + d) + 2n$
  - $(2 + 2c)n + (2d - 3c) = 0$
  - $cn + d$ is a solution iff $2 + 2c = 0, 2d - 3c = 0$
  - $c = -1, d = -3/2,\ a_n^{(p)} = -n - 3/2$
  - $a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n$

Linear Nonhomogeneous Recurrence Relation

- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} + F(n)$, where $c_1, c_2, ..., c_k \in$ R
- $F(n) = (b_t n^t + b_{t-1} n^{t-1} + ... + b_1 n + b_0)s^n$, where $b_0, b_1, ..., b_t \in$ R $s \in$ R
- When s is **not** a root of the characteristic equation
  - $(p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0)s^n$
- When s **is a root of the characteristic equation** and its multiplicity is m, there is a particular solution of the form
  - $n^m(p_t n^t + p_{t-1} n^{t-1} + ... + p_1 n + p_0)s^n$

Case Study

- What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$?

  - The characteristic equation: $r^2 - 6r + 9 = (r-3)^2 = 0$ has a single root, 3, or multiplicity 2.

  - $s = 3$ is a root with multiplicity $m = 2$, but $s = 2$ is not a root.

  - When $F(n) = 3^n$, a particular solution has the form $p_0 n^2 3^n$