Exact Solution of a Minimal Recurrence

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Abstract

In this note we find the exact solution for the minimal recurrence $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k\}$, where a and b are positive integers and $a \geq b$. We prove that the solution is the same as that of the recurrence relation $S_n = aS_{\lceil n/2 \rceil} + bS_{\lfloor n/2 \rfloor}$. In other words, $S_n = S_1 + (a+b-1)S_1 \sum_{i=1}^{n-1} a^{z(i)} b^{\lfloor \lg n \rfloor - z(i)}$, where $z(i)$ is the number of zeros in the binary representation of i. The proof follows from an interesting combinatorial property.

Keyword: analysis of algorithms, computational complexity

1 Introduction

Divide-and-conquer is a very useful method for developing efficient algorithms. The strategy is to partition a problem into several subproblems, find solutions for the parts, and then merge the subsolutions as the final solution for the whole. The time complexity of a divide-and-conquer algorithm can be expressed as a maximin recurrence. To analyze a divide-and-conquer algorithm, we usually need to solve a recurrence. However, it becomes more difficult to obtain the exact solution with the min or max operators in a recurrence, since we cannot simply apply known methods [6] directly. Recurrence of the form $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{S_k + S_{n-k} + f(k)\}$ has been studied by various researchers [1, 5, 2, 7], where f can be any non-decreasing or convex function. All of the previous results have the coefficients of S_k and S_{n-k} as 1.

The so called AND/OR problem [4] is as folows: *Given a box of AND gates, each of fan-in 2, it is known that k OR gates are mistakenly put into a box of AND gates. Suppose that there is no tool to test which are OR gates. In order to compute the AND function of two values correctly, how many gates are necessary to construct an error-free circuit?* This toy problem is for studying fault tolerant computation. A *block circuit* is a leveled circuit and consists of a single gate or three disjoint block subcircuits where the outputs of the two lower-level block subcircuits serve as the input of the top block subcircuit. This is a special leveled circuit model. A typical technique in circuit complexity is restricting the circuit model in order to obtain a better bound when strong bounds under general models are difficult to obtain. When proving the optimality of a *block circuit* design for the AND/OR problem, we solved a minimal recurrence: $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{2S_{n-k} + S_k\}$ [3]. It is crucial to solve the recurrence exactly. We found it difficult to prove the optimality without knowing the exact solution.

By extending the above recurrence, we solve exactly the recurrence $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k\}$, where $a \geq b$. Previously, the exact solutions are known only for the cases $a = b = 1$ and $a = 2, b = 1$. Our proof follows from a crucial and interesting combinatorial property. Since there is no known systematic approach for solving this type of recurrence, we hope shed some light on tackling similar or even more complicated recurrence. To solve exactly the general recurrence $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k + f(k)\}$, where f is any integer function, may not be as easy and new technique may be necessary. We leave it as an open question.

2 Definitions and results

We need some definitions to prove our results.

Definition. $z(i)$ is defined to be the number of zero bits in the binary representation of i.

For example, $z(5) = z(101_2) = 1$. Let k and i be non-negative integers. For any two consecutive integers k and $k+1$, $z(k) \leq 1 + z(k+1)$, since if k is even

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then $z(k) = 1 + z(k+1)$; else $z(k) < 1 + z(k+1)$. For the last inequality, k is odd and thus adding 1 to k will flip the least significant block of 1's into 0's, where the number of zero will not decrease. Similarly, for $0 \le j < 2^i$, it is not hard to see that $z(2^i k + j) \le 1 + z(2^i(k+1) + j)$.

Definition. Let B_1 be any positive integer. For any positive integer $k \ge 2$, B_k is defined as $a^{z(k)} b^{\lfloor \lg k \rfloor - z(k)} B_1$, where a and b are positive integers and $a \ge b$.

For convenience, let $n = 2^i k + j$. Since $z(n) \le 1 + z(n + 2^i)$, we let $d = 1 + z(n + 2^i) - z(n)$. It is clear that d is a non-negative integer. Then $aB_{n+2^i}/bB_n = \frac{a[a^{z(n+2^i)} b^{\lfloor \lg(n+2^i) \rfloor - z(n+2^i)}]}{b[a^{z(n)} b^{\lfloor \lg(n) \rfloor - z(n)}]} = (a/b)^{d+1} b^{\lfloor \lg(n+2^i) \rfloor - \lfloor \lg(n) \rfloor} \ge 1$. In other words, for any positive integer n and non-negative integer i, $aB_{n+2^i} \ge bB_n$. Thus, it is not hard to see the following is true.

Lemma 1 *Let k and i be non-negative integers and $a \ge b$ be positive integers. Then (1) $b \sum_{j=0}^{2^i-1} B_{2^i k+j} \le a \sum_{j=0}^{2^i-1} B_{2^i(k+1)+j}$; (2) For $0 \le \ell < 2^i$, $b \sum_{j=\ell}^{2^i-1} B_{2^i k+j} + b \sum_{j=0}^{\ell-1} B_{2^i(k+1)+j} \le a \sum_{j=\ell}^{2^i-1} B_{2^i(k+1)+j} + a \sum_{j=0}^{\ell-1} B_{2^i(k+2)+j}$.*

The above lemma shows that for any two consecutive blocks of B_i's (each of size 2^i) the inequality holds. We prove that the inequality actually holds for any two equal-sized consecutive blocks of B_i's in the following lemma.

Lemma 2 *Let Δ be a positive integer such that $2^{i-1} \le \Delta < 2^i$ for some positive integer i. For any positive integer $n > \Delta$, $b \sum_{j=0}^{\Delta-1} B_{n-\Delta+j} \le a \sum_{j=0}^{\Delta-1} B_{n+j}$.*

Proof.

We prove by induction on i. It is clear for $i = 1$. Suppose that the claim is true for smaller cases up to $i-1$, i.e. up to the case of $2^{i-2} \le \Delta < 2^{i-1}$. Now we want to prove the case for $2^{i-1} \le \Delta < 2^i$. For any two consecutive blocks of B_i's, each of size Δ, there is an integer n such that these two blocks can be written as $B_{n-\Delta}, B_{n-\Delta+1}, \cdots, B_{n-1}$ and $B_n, B_{n+1}, \cdots, B_{n+\Delta-1}$. It is clear that for $j = 0, \cdots, (2\Delta - 2^i - 1)$, $bB_{n-\Delta+j} \le aB_{n-\Delta+2^i+j}$. Now the problem is reduced to check if the relation holds for the last $2^i - \Delta$ B_j's of the first block and the first $2^i - \Delta$ B_j's of the second block. In other words, we are checking the relation between two consecutive blocks of B_j's, where each block has size $2^i - \Delta \le 2^{i-1}$. This is true by induction hypothesis and lemma 1 for Δ as a power of 2. This completes our proof. □

Similarly, by induction, we can prove the following lemma, where the two blocks of B_i's are not consecutive.

Lemma 3 *Let Δ be a positive integer such that $2^{i-1} \le \Delta < 2^i$ for some positive integer i. For any positive integer $n > \Delta$, $b \sum_{j=0}^{\Delta-1} B_{n-\Delta+j} \le a \sum_{j=1}^{\Delta} B_{n+j}$.*

Proof. Note that B_n doesn't belong to these two blocks. First we prove the cases for Δ as a power of 2. Let $\Delta = 2^i$, where i can be any non-negative integer. Then it is clear that these two blocks consist of the followings: $B_{n-2^i}, B_{n-2^i+1}, \cdots, B_{n-1}$, and $B_{n+1}, \cdots, B_{n+2^i}$. It is clear that for $j = 1, \cdots, 2^i - 1$, $bB_{n-2^i+j} \le aB_{n+j}$. Also, it is clear that $bB_{n-2^i} \le aB_{n+2^i}$. Thus, our claim is true for Δ as a power of 2.

Next we are to prove the cases when Δ is not exactly a power of 2. We prove by induction on i. It is clear for $i = 1$. As above we suppose that the claim is true for smaller i, i.e. for the cases up to $2^{i-2} \le \Delta < 2^{i-1}$. Now we want to prove the case for $2^{i-1} \le \Delta < 2^i$. The two blocks as stated in the claim can be illustrated as: $B_{n-\Delta}, B_{n-\Delta+1}, \cdots, B_{n-1}$, and $B_{n+1}, \cdots, B_{n+\Delta}$, where B_n is not in the blocks. It is clear that for $j = 0, \cdots, (2\Delta - 2^i)$, $bB_{n-\Delta+j} \le aB_{n-\Delta+2^i+j}$. Now the problem is reduced to check if the relation holds for the last $2^i - \Delta - 1$ B_j's of the first block and the first $2^i - \Delta - 1$ B_j's of the second block. In other words, we are checking the relation between two smaller blocks of B_j's, where each block has size $2^i - \Delta - 1 \le 2^{i-1}$. While this is true by induction hypothesis and the proof for Δ as a power of 2. This completes our proof. □

Next we solve the recurrence relation $S_n = aS_{\lceil n/2 \rceil} + bS_{\lfloor n/2 \rfloor}$, whose solution will be used to solve the minimal recurrence relation.

Lemma 4 *Let a and b be positive integers and $a \ge b$. The recurrence relation $S_n = aS_{\lceil n/2 \rceil} + bS_{\lfloor n/2 \rfloor}$ has the solution $S_n = S_1 + (a+b-1)S_1 \sum_{i=1}^{n-1} a^{z(i)} b^{\lfloor \lg i \rfloor - z(i)}$, where $z(i)$ is the number of zero in the binary representation of i.*

Proof. Let $D_n = S_{n+1} - S_n$. Then $D_{2n} = S_{2n+1} - S_{2n} = (aS_{n+1} + bS_n) - (a+b)S_n = a(S_{n+1} - S_n) = aD_n$. $D_{2n+1} = S_{2n+2} - S_{2n+1} = (a+b)S_{n+1} - (aS_{n+1} + bS_n) = b(S_{n+1} - S_n) = bD_n$. Note that D_n can be determined by checking if it is even inductively by checking the bits in the binary representation of n. Thus we have $D_n = a^{z(n)} b^{\lfloor \lg n \rfloor - z(n)} D_1$ and $S_n = S_1 + \sum_{i=1}^{n-1} D_i = S_1 + (a+b-1)S_1 \sum_{i=1}^{n-1} a^{z(i)} b^{\lfloor \lg i \rfloor - z(i)}$. It is clear that $D_1 = S_2 - S_1 = (a+b)S_1 - S_1 = (a+b-1)S_1$. Note that D_i is the same as B_i, if we let $B_1 = D_1$. □

Next we prove that the above solution actually achieves the optimality for $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k\}$. I.e. the minimum happens when $k = \lfloor n/2 \rfloor$.

Theorem 5 *$S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k\}$ and $S_n = aS_{\lceil n/2 \rceil} + bS_{\lfloor n/2 \rfloor}$ have the same solution.*

Proof. It is clear for $n = 2$. By induction, suppose it is true for the cases less than n. Let $k < \lfloor n/2 \rfloor$. Then $n - k > n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$.

$$
\begin{aligned}
& aS_{n-k} + bS_k - aS_{\lceil n/2 \rceil} - bS_{\lfloor n/2 \rfloor} \\
= \ & a(S_{n-k} - S_{\lceil n/2 \rceil}) - b(S_{\lfloor n/2 \rfloor} - S_k) \\
= \ & a \sum_{i=\lceil n/2 \rceil}^{n-k-1} B_i - b \sum_{i=k}^{\lfloor n/2 \rfloor - 1} B_i \\
\geq \ & 0.
\end{aligned}
$$

The inequality follows from the previous two lemmas, i.e. if n is even, it follows from Lemma 2; otherwise by Lemma 3. The above shows that the minimum of the first recurrence relation occurs when $k = \lfloor n/2 \rfloor$. This completes the proof. □

3 Conclusion and remarks

We have solved exactly the recurrence $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k\}$. This type of recurrence does happen in analyzing algorithms and circuit designs. Our proof does not apply to the case when $a < b$. Also the range in the summation of the recurrence cannot go beyond $\lfloor n/2 \rfloor$. For these cases, new technique may be needed. One interesting open question is: Is it possible to extend our method to solve exactly the recurrence $S_n = \min_{k=1}^{\lfloor n/2 \rfloor}\{aS_{n-k} + bS_k + f(k)\}$?

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