On the Fairness and Complexity of Generalized k-in-a-row Games

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Abstract

Recently, Wu and Huang[14] introduced a new game called **Connect6**, where two players, Black and White, alternately place two stones of their own color, black and white respectively, on an empty Go-like board, except for that Black (the first player) places one stone only for the first move. The one who gets six consecutive (horizontally, vertically or diagonally) stones of his color first wins the game. Unlike Go-Moku, **Connect6** appears to be fairer and has been adopted as an official competition event in Computer Olympiad 2006.

$Connect(m, n, k, p, q)$ is a generalized family of k-in-a-row games, where two players place p stones on an $m \times n$ board alternatively, except Black places q stones in the first move. The one who first gets his stones k-consecutive in a line (horizontally, vertically or diagonally) wins. **Connect6** is simply the game of $Connect(m, n, 6, 2, 1)$. In this paper, we study two interesting issues of $Connect(m, n, k, p, q)$: fairness and complexity. First, we prove that no one has a winning strategy in $Connect(m, n, k, p, q)$ starting from an empty board when $k \geq 4p + 7$ and $p \geq q$. Second, we prove that, for any fixed constants k, p such that $k - p \geq \max\{3, p\}$ and a given $Connect(m, n, k, p, q)$ position, it is PSPACE-complete to determine whether the first player has a winning strategy. Consequently, this implies that the **Connect6** played on an $m \times n$ board (i.e., $Connect(m, n, 6, 2, 1)$) is PSPACE-complete.

1 Introduction and preliminaries

The game family *k-in-a-row*, or (m, n, k)-Games, is well-known and has been studied for a while. It is a two-player game played on an $m \times n$ board. Two players P_1 and P_2 alternatively place one black and one white stone, respectively, on an unoccupied square on the board and the one who first gets his stones k-consecutive in a line (horizontally, vertically or diagonally) wins. Some of the special (m, n, k)-games, such as *TicTacToe* ($(3, 3, 3)$-game) and *Go-Moku* ($(19, 19, 5)$-game), are very popular worldwide. Moreover, there are many other

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modified versions, such as *Maker-Breaker version*, *Inverse version*, *Periodic version*, *Higher dimensions version*, *Multi-Player version* and so on[5, 9, 15]. Maker-Breaker is an asymmetric version of the extended (m, n, k)-Game, where in each move, Maker(P_1) or Breaker(P_2) can place $t \geq 1$ stones, and Maker wins by getting his stones k-consecutive in a line, while Breaker wins by preventing Maker from winning. The winning condition of Inverse k-in-a-row game (or commonly called Misère game) is opposite to that of k-in-a-row game: The one who gets his stones k-in-a-row first loses the game. This version is obviously more complex, since the goal of each player is to avoid his stones k-in-a-row and it is hard to force opponent's stones k-in-a-row. The board of Periodic k-in-a-row game has some modifications: the left connects with the right, and the top connects with the bottom. As for the Higher dimensions version and Multi-Player version, it just extends the board into three or more dimensions and extends two players to three or more players, respectively. Such modified versions are interesting and generally studied in theoretical computer science and combinatorial game theory.

In this paper, we study a new family of generalized k-in-a-row games: $Connect(m, n, k, p, q)$, which was introduced by Wu and Huang [14]. In $Connect(m, n, k, p, q)$, two players place p stones on an $m \times n$ board alternatively, except Black places q stones in the first step. The one who first gets his stones k-consecutive in a line (horizontally, vertically or diagonally) wins. For example, TicTacToe is a $Connect(3, 3, 3, 1, 1)$ game, Go-Moku is a $Connect(19, 19, 5, 1, 1)$ game and **Connect6** is simply a $Connect(m, n, 6, 2, 1)$ game. For convenience, $Connect(\infty, \infty, k, p, q)$ is denoted as $Connect(k, p, q)$[14]. W.L.O.G., we can assume that $\max\{m, n\} \geq k > p \geq q > 0$. Recently, **Connect6** has become an official competition event in the 11th Computer Olympiad in 2006 because of its fairness and state-space complexity. For further fairness and state-space complexity discussion of **Connect6**, we refer to Wu and Huang's paper[14].

Talking about (the generalized) k-in-a-row games, fairness is considered to be the most interesting issue. Herik et al.[13] gave a definition of fairness as follows.

Definition 1 [13] *A game is fair if (1) this game has a draw for two perfect players and (2) both players have a roughly equal probability on making a mistake.*

However, it is hard to prove that whether both players have a roughly equal probability on making a mistake in general. Wu and Huang[14] gave some empirical results for small cases with $k \leq 9$ and $k - p \leq 3$. For convenience, this paper focuses on the first part of fairness defined above, i.e., for two perfect players, whether $Connect(m, n, k, p, q)$ has a draw or who can win? There are some partial results for this question. For example, the *strategy-stealing argument* shows that P_2 has no winning strategy when $q \geq p$: Suppose P_2 has a winning strategy, then P_1 can make the first move randomly, act as the second player and win by stealing P_2's strategy, which leads to a contradiction. Moreover, for $p = q = 1$, Herik et al.[13] listed resolved cases for $k \leq 5$, while Zetters [16] proved that P_2 can tie the infinite game when $k \geq 8$ and as a consequence on any finite board. The cases when $k = 6, 7$ are still unknown. As for the results of the generalized k-in-a-row games, Wu and Huang[14] showed that P_1 can win $Connect(k, p, q)$ when $p < \lfloor \frac{q}{\delta^2} \rfloor (4\delta + 4) + \min(q$

mod $\delta^2, \lfloor \frac{8q}{\delta^2} \rfloor$), where $\delta = k - p$. Pluhár[9] showed that P_2 can tie $Connect(k, p, q)$ when $k \geq p + 80 \log_2 p + 160$ and $q = p \geq 1000$, and as a consequence that no one has a winning strategy in $Connect(m, n, k, p, q)$ for any m, n when $k \geq p + 80 \log_2 p + 160$ and $p \geq q$. However, this bound can be large even for small p. In section 2, we give a better result for small p. Indeed, we prove that no one has a winning strategy in $Connect(m, n, k, p, q)$ for any m, n when $k \geq 4p + 7$ and $p \geq q$. As a result, our bound is better than Pluhár's bound for smaller $p(\leq 265)$, although their result is asymptotically better.

Another important issue, for mathematical games, is complexity. The hardness of many popular "small" games is not as easy as we think. Furthermore, two-player games are often more complicated than one-player games. For example, it is shown to be PSPACE-complete for Go-moku[10] and Othello[1], EXPTIME-complete for Checkers[11], while NP-complete for Minesweeper[3]. For further readings, we refer to Nowakowski's books[6, 7]. In section 3, we study the complexity of $Connect(m, n, k, p, q)$.

Definition 2 [4, 8, 12] *A problem is said to be PSPACE-complete if it can be solved within polynomial space and every problem solvable in polynomial space can be reduced to it in polynomial time. (Note that polynomial space/time mentioned here is with respect to the input size.)*

Definition 3 *For any fixed constants k, p and given an arbitrary $Connect(m, n, k, p, q)$ game position, the decision version of the $Connect(m, n, k, p, q)$ problem is to determine whether P_1 has a winning strategy.*

We will prove that the decision $Connect(m, n, k, p, q)$ problem is PSPACE-complete for $k - p \geq \max\{3, p\}$ by reducing *the generalized geography* game played on a planar bipartite graph of maximum degree 3 to it. The generalized geography game is a two-player game ($\exists$-player and $\forall$-player) on a directed graph. The $\exists$-player starts from a specific marked starting vertex, and both players alternatively mark any unmarked vertex to which there is an arc from the last marked vertex. The one who cannot mark a vertex anymore loses the game. Sipser[12] showed that the generalized geography game is PSPACE-complete.

Theorem 1 [4, 8, 12] *Generalized geography played on a planar bipartite graph of maximum degree 3 is PSPACE-complete.*

A key idea of the reduction is to "embed" a graph into the connect game board, and thus two players will be forced to play the geography game in effect. To ensure the embedding can be done in polynomial time, the following theorem is useful.

Theorem 2 [2] *There is a linear time algorithm to draw any planar graph of maximum degree 3 on a $\lfloor \frac{V}{2} \rfloor \times \lfloor \frac{V}{2} \rfloor$ grid orthogonally, where V is the number of vertices. Moreover, each edge has at most 1 bend.*

Once the instance of the geography game is drawn orthogonally on a grid, we can transform it into a connect game board efficiently. We will show the details of the reduction in Section 3. We also need the following definition of *threat* in a game.

Definition 4 [14] *In a Connect(m, n, k, p, q) game, a player is said to have t threats, if and only if his opponent needs to place t stones to prevent him from winning on his next move.*

There are other interesting implementation issues, such as game strategy, search technique and so on, which are typically addressed in artificial intelligence. Wu and Huang's paper[14] shows some related results and useful references. In this paper, we focus on the theoretical foundation of the games.

2 Fairness

Since we have shown that P_2 has no winning strategy when $q \geq p$, by the strategy-stealing argument, we focus on the cases when $q \leq p$ in this section. The following is our first result.

Theorem 3 *No one has a winning strategy in Connect(m, n, k, p, q) for any m, n with $\max\{m, n\} \geq k$ when $q \leq p$ and $k \geq 4p + 7$.*

To prove Theorem 3, we define a new game modified from the Maker-Breaker game, denoted as $mMB(n, p)$ for short, which is a two-player game played on an $n \times n$ board. In move $2i - 1$, $i \in N$, P_1 can choose an integer t, $1 \leq t \leq p$, and then P_1 and P_2 are required to place exactly t black stones and t white stones in move $2i - 1$ and move $2i$, respectively, until there is a winner or no more empty square. If there exist n black stones in a line (horizontally or vertically, but not diagonally), then P_1 wins, else P_2 wins. Since it can be easily verified that P_1 wins when $p \geq n - 1$ (i.e., P_1 can pick $t = 1$ on his first move and then pick $t = n-1$ and place $n-1$ black stone to have n black stones in a line on his second move), we can assume $p \leq n - 2$. In the following, we will focus on the $mMB(n, p)$ game, and our goal is to prove that P_2 has a winning strategy, i.e., preventing P_1 from winning. For convenience, we define environment variables r_i for the i-th row, $1 \leq i \leq n$. The value of r_i is equal to -1 if there exists a white stone in the i-th row; otherwise, r_i indicates the number of black stones in the i-th row. The environment variables c_j for the j-th column, $1 \leq j \leq n$, are defined similarly, and we let $R = \{r_i | 1 \leq i \leq n\}$ and $C = \{c_i | 1 \leq i \leq n\}$. Moreover, we use $(x, y) = B$ and W to denote that square (x, y) has a black stone and a white stone respectively, and E for empty. We will use a "loop invariant" as shown in Lemma 1 to prove that P_1 cannot win, where a "loop" consists of one move of P_1 and the countermove of P_2. We call $C \cup R$ *safe* if (1) P_1 cannot win, and (2) there is at most one variable in $C \cup R$ that is positive and no variable is greater than 1.

Lemma 1 *In an $mMB(p + 2, p)$ game position, assume $C \cup R$ is safe. Then for each move by P_1, there is a move by P_2 such that $C \cup R$ is safe.*

Proof. Since at most one variable is positive, say $c_i = 1$, there are at most $p + 1$ black stones in a line after P_1's move. Note that P_1 wins if and only if there are $p + 2$ black stones in a column or row. We prove the rest by induction on the integer t that P_1 chooses. P_2's

response is given in the induction step below.

Basis: ($t = 1$) By assumption, there is at most one variable c_i with value 1. Assume P_1 places one black stone at (a, b). First, we consider the case when $b \neq i$ and thus $r_a \leq 1$, $c_b \leq 1$, $c_i \leq 1$, since there may be white stone(s) in row a or column b. If $(a, i) = W$ or $(a, i) = E$ in which case P_2 can place a white stone at (a, i), then we have $r_a = -1$, $c_b \leq 1$, $c_i = -1$ and the lemma holds. Suppose $(a, i) = B$, then $r_a = -1$ by the assumption that there is only one variable $c_i = 1$. Hence if there is already one white stone in the i-th column or P_2 places a white stone at any empty square in the i-th column, then we are done with at most one variable $c_b = 1$. Next, we consider the case when $b = i$ and then we have $r_a \leq 1$ and $c_i \leq 2$. Since there must be an empty square in column i, P_2 can place a white stone in the i-th column, and then this lemma holds with at most one variable $r_a = 1$.

Induction: Assume it is true for all t up to $w < p$. Consider the case $t = w + 1$. By the hypothesis, P_2 has a response S_w using w white stones against the first w black stones P_1 placed by ignoring the existence of the $(w + 1)$-st black stone. Assume P_1 placed the $(w+1)$-st black stone at (a, b). If S_w doesn't place a white stone at (a, b), then it reduces to the case $t = 1$. If S_w chooses (a, b), we know that there are at most three variables with positive values (i.e. $r_a > 0$, $c_b > 0$, $c_i \leq 1$), and P_2 still has two white stones to play (i.e., the one placed at (a, b) is withdrawn). Then P_2 can place the two white stones in the a-th row and the b-th column, and we are done with $c_i \leq 1$. □

Lemma 2 *P_2 has a winning strategy in $mMB(p + 2, p)$.*

Proof. Since all variables in $C \cup R$ are zero in the beginning, by Lemma 1, we know that P_2 has a winning strategy. □

Then we have the following obvious consequence.

Corollary 1 *P_2 can win $mMB(n, p)$, when $n \geq p + 2$.*

Lemma 3 *P_2 can tie $Connect(k, p, q)$ when $q \leq p$ and $k \geq 4p + 7$.*

Proof. The strategy for P_2 is by divide-and-conquer:

1. Tile the game board with infinite many $(p+2) \times (p+2)$ tiles as shown in Figure 2 (an example of $p + 2 = 4$). There are three types of tiles: A, B and C as shown in Figure 1. Tile B and C are just "twisted" from tile A.

2. In each tile where P_1 placed black stones, P_2 responds with the same number of white stones in it. This is possible since $q \leq p$. If $q < p$, P_2 has an extra $p - q$ white stones to play in the following move.

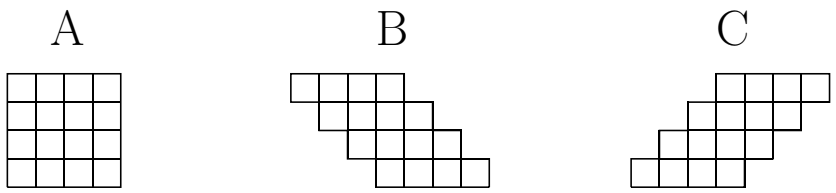


Figure 1: Three types of tiles of size $(p+2) \times (p+2)$, for $p = 2$.

3. Play $mMB(p+2, p)$ game in each tile, where P_2 has enough white stones to play. Note that when playing in a twisted tile P_1 tries to get $(p+2)$ black stones horizontally or diagonally. By Lemma 2, we know that P_2 has a strategy such that there are at most $p+1$ black stones in a line in each tile A(horizontally or vertically), at most $p+1$ black stones in a line in each tile B(horizontally or diagonally down), and at most $p+1$ black stones in a line in each tile C(horizontally or diagonally up).

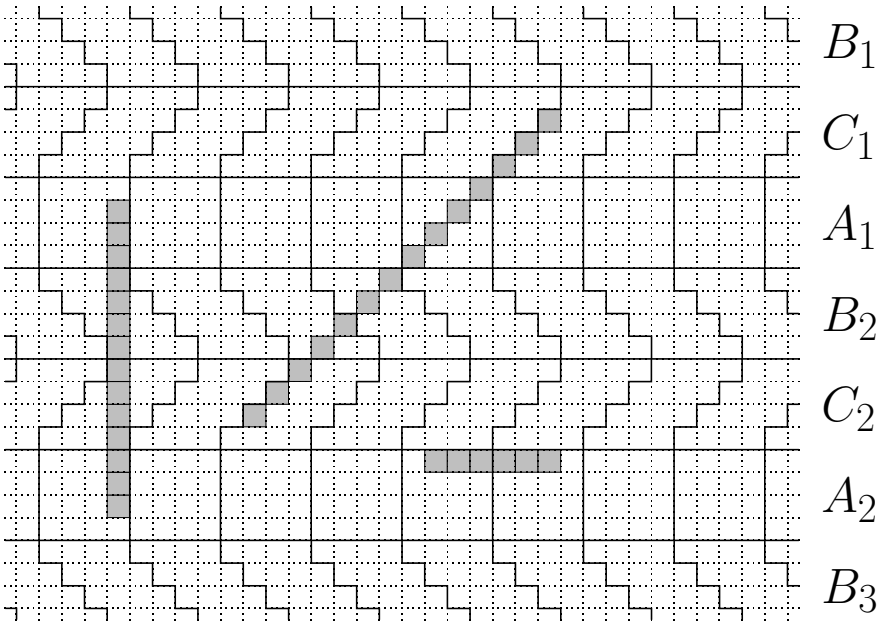


Figure 2: The game board is divided into infinite many 4×4 tiles.

In the whole game board (refer to Figure 2), since there are at most $(p+1)$-consecutive black stones in a vertical line in section $\{A_i\}$ and at most $(p+2)$-consecutive black stones in a vertical line in section $\{B_i, C_i\}$, we obtain that there are at most $(4p+6)$-consecutive black stones in a vertical line, i.e., $(p+1)$ in A_1, $(p+2)$ in B_2, $(p+2)$ in C_2, and $(p+1)$ in A_2. Similarly, there are also at most $(4p+6)$-consecutive black stones in a diagonal line (diagonally up and diagonally down). As for the horizontal line, since there are at most $(p+1)$ black stones in section $\{A_i, B_i, C_i\}$, there are at most $(2p+2)$-consecutive black stones in a horizontal line. We show the longest possible black lines with the shaded cells in Figure

2. From the above, we get the desired result: there are at most $(4p+6)$-consecutive black stones in a line (horizontally, vertically and diagonally). □

Corollary 2 *P_2 can tie $Connect(m, n, k, p, q)$ for any m, n when $q \leq p$ and $k \geq 4p + 7$.*

Proof. It is obvious that the strategy for P_2 shown in the proof of Lemma 3 can be applied to any finite board. □

Proof of Theorem 3. This is true since P_1 can adopt the strategy for P_2, shown in Lemma 3 as well. □

3 PSPACE-completeness

In this section, we investigate the computational complexity of the decision $Connect(m, n, k, p, q)$ problem. Recall that the decision $Connect(m, n, k, p, q)$ problem is to determine whether P_1 has a winning strategy when given an arbitrary non-empty $Connect(m, n, k, p, q)$ position, where k and p are fixed constants. Since the given game position in the decision problem is not an empty board, q is irrelevant. We will show that the decision $Connect(m, n, k, p, q)$ problem is PSPACE-complete when $k - p \geq \max\{3, p\}$.

Lemma 4 *The decision $Connect(m, n, k, p, q)$ problem is in PSPACE.*

Proof. Since this game must end in $O(mn)$ steps, this problem can be computed by an alternating Turing machince in polynomial time. We know that ATIME(poly) = PSPACE [12]. Hence, this problem is in PSPACE. □

The next step is to show the PSPACE-hardness of the decision $Connect(m, n, k, p, q)$ problem. It is already known for the case $p = 1$.

Lemma 5 [10] *The decision $Connect(m, n, k, p, q)$ problem is PSPACE-hard when $k \geq 5$ and $p = 1$.*

We focus on the case $p \geq 2$ as follows. We show a polynomial time reduction from the generalized geography game played on a planar bipartite graph of maximum degree 3 to the decision $Connect(m, n, k, p, q)$ problem. For an arbitrary generalized geography game, we will construct a corresponding $Connect(m, n, k, p, q)$ game position, where m, n are polynomial in terms of the input size and q is negligible, such that the $\exists$-player has a winning strategy in the generalized geography game if and only if P_1 has a winning strategy from the constructed game position.

The main difference between Go-Moku and $Connect(m, n, k, p, q)$ is that each player can place more than one stone in a move in $Connect(m, n, k, p, q)$ game. In order to deal with this difference, we construct the connect game position with one simulation zone, one winning zone and $p - 1$ auxiliary zones, as shown in Figure 3. The key idea behind the construction

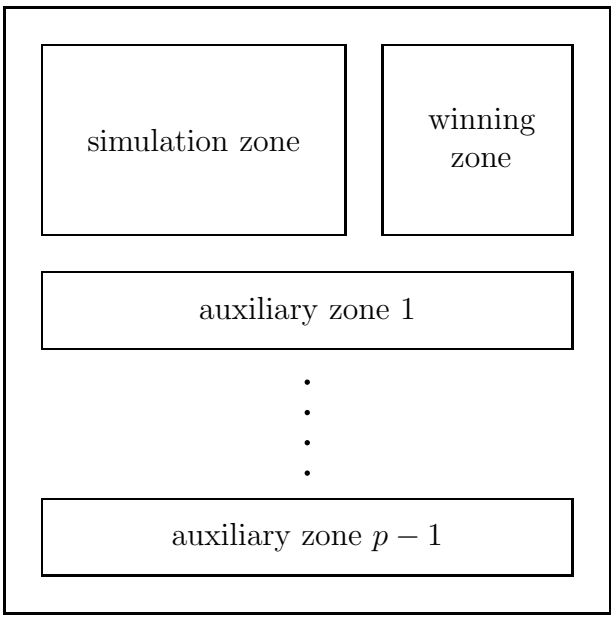


Figure 3: The global view of the constructed connect game position.

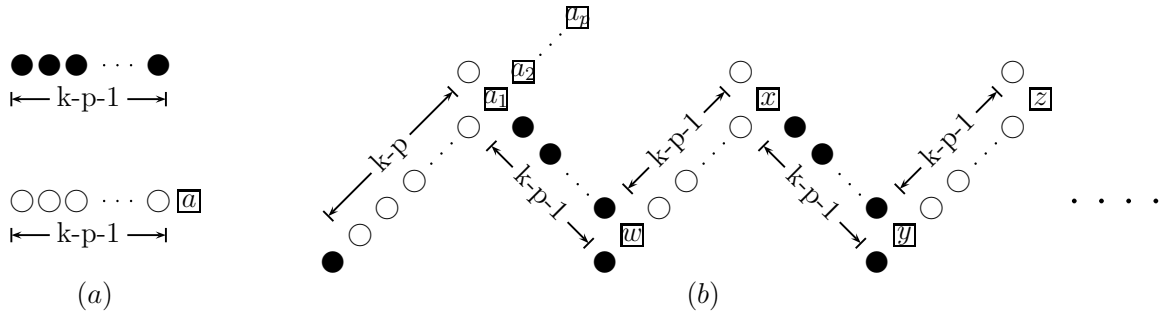
is to force each player to place exactly one stone in each of the $p-1$ auxiliary zones and the simulation zone until the play in the simulation zone terminates, which means no more stone will be placed in the simulation zone. The constructed position in the simulation zone, in effect, forces P_1 and P_2 to play the generalized geography game. The winner in the simulation zone, can then place stones in the winning zone and will win. Next, we show the constructed position of each zone in details.

Construction of winning zone and auxiliary zones:

The constructed position in the winning zone is shown in Figure 4(a) and the constructed position in an auxiliary zone is shown in Figure 4(b). Note that the constructed positions of the $p-1$ auxiliary zones are the same and the number of the repeated patterns will be determined later. In Figure 4, we find that no one has threat in the winning zone (there are only $k-p-1$ black and white stones, respectively), and P_2 has exactly one threat in each auxiliary zone (the $(k-p)$-consecutive white stones on the left-hand side), hence $p-1$ threats in total, while P_1 has no threat in any of the auxiliary zones yet.

Construction of simulation zone:

We construct the simulation zone from an instance of the geography game. The purpose is to force two players to play the generalized geography game in the simulation zone. For a planar bipartite geography graph with maximum degree 3, we will first apply Theorem 2 to draw it orthogonally on a grid, and then construct a corresponding game position in the



(a) (b)

Figure 4: (a) The constructed position in the winning zone. (b) The constructed position in an auxiliary zone.

simulation zone. Next, we give the corresponding constructed positions of the vertices and arcs, which are represented as gadgets. We use the gadgets to construct the desired connect position. Since the vertices and arcs are represented as gadgets, we can use copies of their mirror images or rotate them 90^o, 180^o or 270^o whenever necessary .

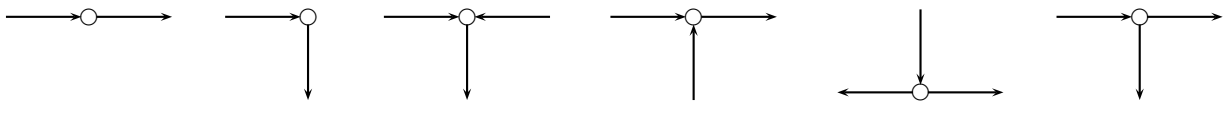
In the construction, each vertex has one or more *entry* points and *exit* points (since there are 3 types of vertices to be defined later) and each arc has a *head* point and a *tail* point. More specifically, for an arc (u, v), the head point of its corresponding gadget will connect (overlap) the corresponding exit point of u's gadget, and its gadget tail point will connect (overlap) the corresponding entry point of v's gadget. We show the constructions of gadgets as follows.

Gadgets for vertices:

Since the $Connect(m, n, k, p, q)$ game is a two-player game and the generalized geography game is played on a bipartite graph, we can divide all vertices into two groups, Black and White. Moreover, the starting vertex belongs to Black, and w.l.o.g., we will illustrate the constructed positions of the vertices in the Black group as in the following examples. The constructed positions of the vertices in the White group can be obtained by exchanging the colors. According to [4, 8], there are only three types of vertices in the generalized geography game played on a bipartite graph with maximum degree 3, i.e., (1) in-degree 1 and out-degree 1, (2) in-degree 2 and out-degree 1, (3) in-degree 1 and out-degree 2, and for each we construct two kinds of positions as shown in Figure 5. The constructed positions of the six kinds of vertices are shown in Figures 6 to 8. In Figure 6, the entry point is at a and the exit point is at h. In Figure 7, the two entry points are at a and e, and the exit point is at z. In Figure 8, the entry point is at a and the two exit points are at u and z.

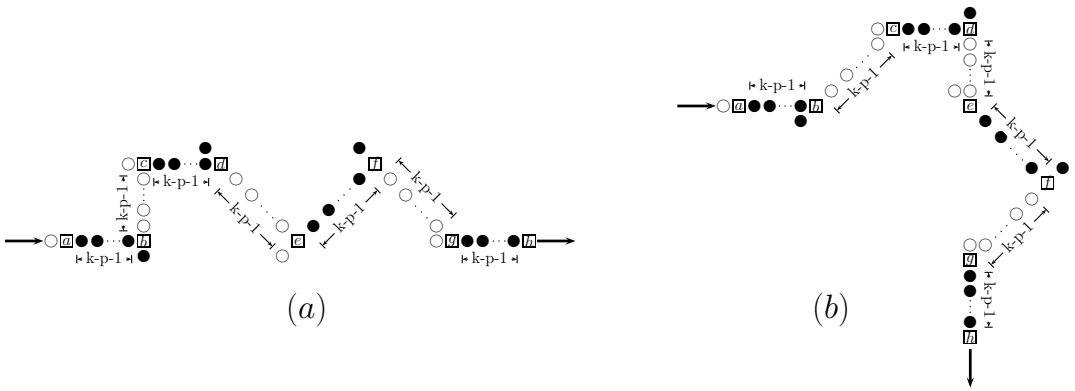
Gadgets for arcs:

There are two types of arcs: (1) from a vertex in the Black group to a vertex in the White group, and (2) from the White group to the Black group. W.L.O.G., we show the



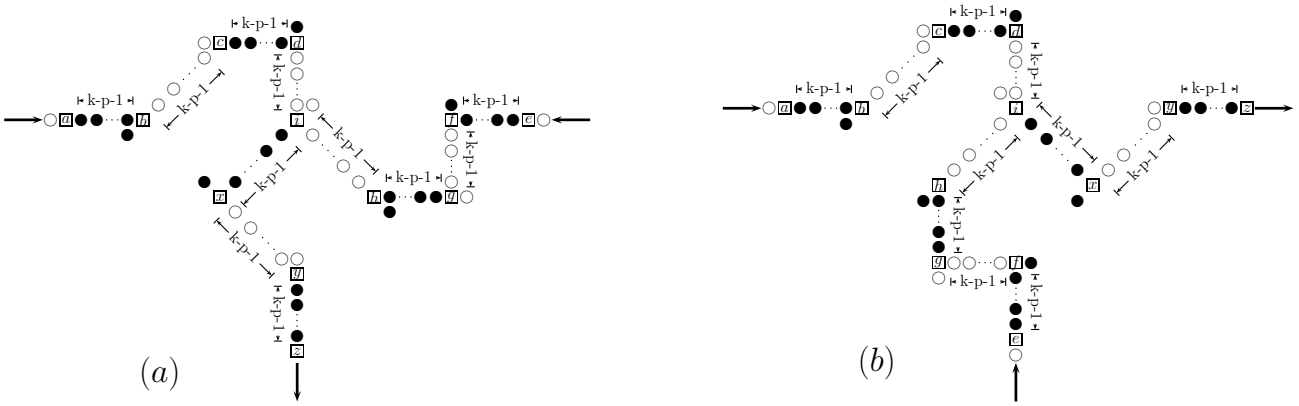
$(1a)$ $(1b)$ $(2a)$ $(2b)$ $(3a)$ $(3b)$

Figure 5: $(1a)$ and $(1b)$ are two kinds of vertices with in-degree and out-degree 1. $(2a)$ and $(2b)$ are two kinds of vertices with in-degree 2 and out-degree 1. $(3a)$ and $(3b)$ are two kinds of vertices with in-degree 1 and out-degree 2. The other possibilities can be obtained by flipping or rotating the above.



(a) (b)

Figure 6: Vertices with in-degree 1 and out-degree 1 correspond to Figure 5$(1a)$ and $(1b)$, where node a indicates the entry point and node h the exit point.



(a) (b)

Figure 7: Vertices with in-degree 2 and out-degree 1 correspond to Figure 5$(2a)$ and $(2b)$, where nodes a and e indicates the entry points and z for the exit point.

construction of type-1 arcs, and type-2 arcs can be obtained by exchanging the colors of the stones. By Theorem 2, we can assume that each arc is horizontal, vertical, or composed of a horizontal segment and a vertical segment. The constructed position of the bend of an

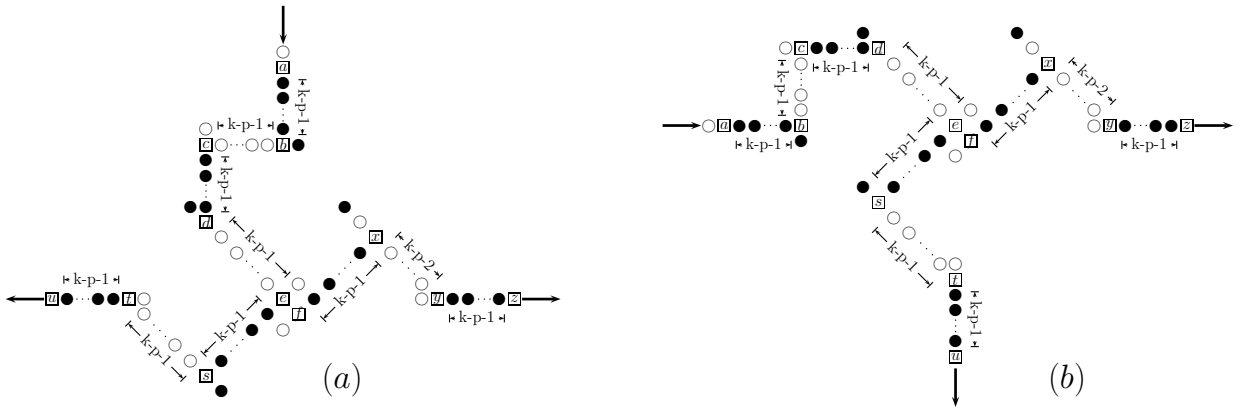


Figure 8: Vertices with in-degree 1 and out-degree 2 correspond to Figure 5(3a) and (3b), where nodes u and z indicate the exit points and a for the entry point.

arc is shown in Figure 9(c). Since the usage of a bend is to connect a vertical segment and a horizontal segment, we can view it as a special vertex with the entry point at a and exit point at i. Furthermore, each straight arc and straight segment may consist of several unit components as shown in Figure 9(a), and we define the distance between a and c as a unit for convenience (that is, each unit equals $2(k-p)$ in the game board and suppose each side of the cell in the game board has length 1). Thus each arc gadget is of integer unit of length. Moreover, the head point of the arc shown in Figure 9(b) is at a and the tail point is at e.

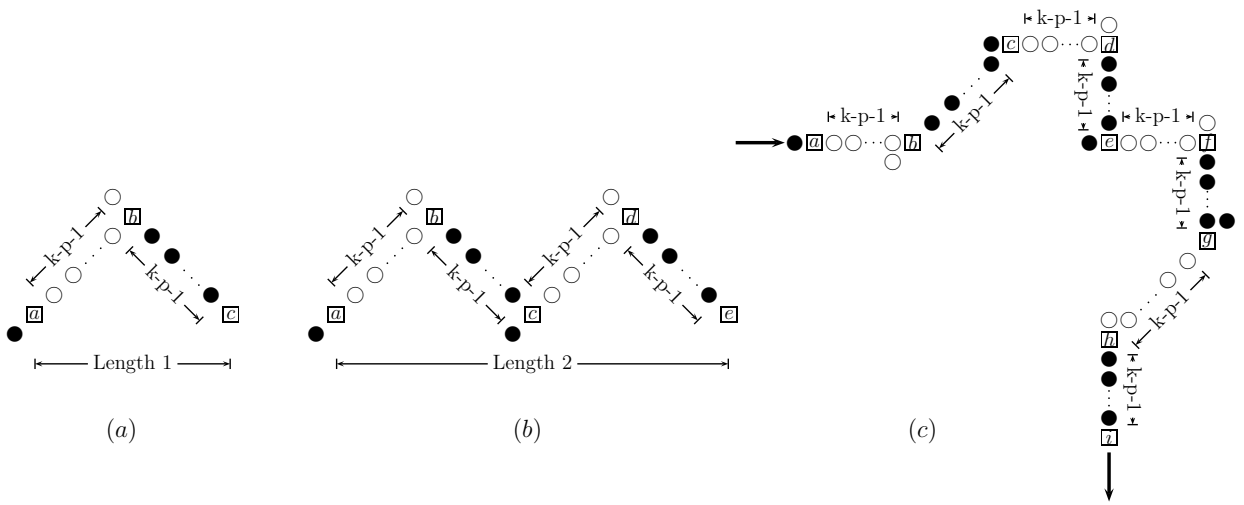


Figure 9: (a) A unit component. (b) An arc with 2 units of length. (c) Gadget for a bend.

Put it together:

To obtain the desired connect position in the simulation zone, we need to deal with: (1) the starting vertex and (2) embedding the geography graph into the game board correctly. First the construction of the starting vertex is easy. Since there are only six kinds of vertices, we have six kinds of starting vertices. The constructed position of a starting vertex with in-degree 1 and out-degree 1, as shown in Figure 10, is modified from Figure 6(a). The location a is replaced with a black stone and location b with a white stone. The constructed positions of the starting vertex of the other five kinds can be obtained similarly as follows. In Figure 6(b), replace a with a black stone and b with a white stone. In Figure 7, replace a, e, c, g with black stones and b, f, d, h with white stones. In Figure 8, replace a with a black stone and b with a white stone. Note that, in the simulation zone, P_2 has exactly one threat in the starting vertex (refer to Figure 10 for example) while P_1 has no threat.

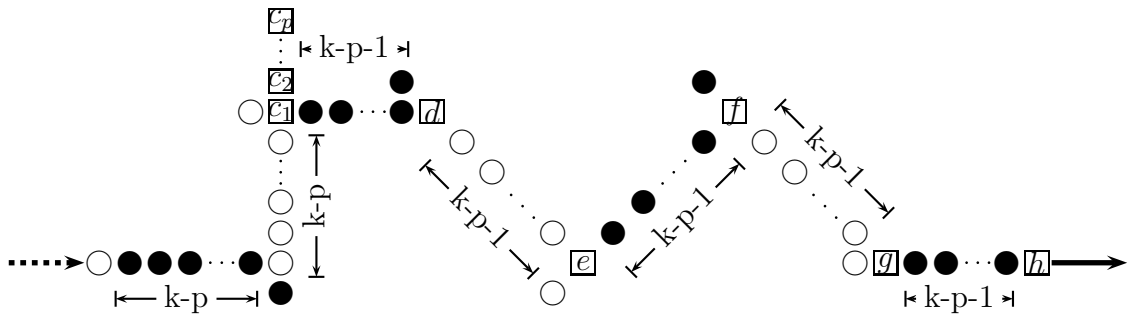


Figure 10: A starting vertex with in-degree 1 and out-degree 1 corresponds to Figure 5.

Second, we need to embed the geography graph into the game board. By Theorem 2, we can assume the geography graph is drawn orthogonally on a $\lfloor \frac{V}{2} \rfloor \times \lfloor \frac{V}{2} \rfloor$ grid, where V is the number of vertices in the geography graph. Next, we map (i, j) in the grid to $(t \times L \times i + 3 \times L, t \times L \times j + 3 \times L)$ in the game board, where $L = 2(k - p)$ and t is a large enough constant (for instance, 5 is enough). The reason why we add $3 \times L$ is to reserve spaces for the boundaries.

The constructed position of each vertex has a *center point*, and we will put the center point at the corresponding coordinate of the vertex. Note that the bend of an arc is viewed as a special vertex. Both of the center points in Figure 6 (Figure 7 and Figure 8) are at e (i and e respectively). The center point of a bend as in Figure 9(c) is at e. In Figure 6(a) and 6(b), the distance between the center point and the entry (exit) point has $\frac{3}{2}$ units of length. Moreover, the entry (exit) point and the center point is either in the same vertical or horizontal line. The same properties also hold in Figure 7, Figure 8 and Figure 9(c).

Now we are going to connect the head (tail) point of an arc gadget with the exit (entry respectively) point of the corresponding vertex gadget. Since each arc is of integer units of length, and its corresponding entry point and exit point lie in a line (we can view the bend as a vertex here) with distance a multiple of unit length, we can connect the vertex gadgets and the arc gadgets correctly.

An example with $k-p=4$ (ignoring the boundaries) is shown in Figure 11. The starting vertex is located at (0,0). The vertices located at {(0,0), (1,1), (0,2)} belong to the Black group, and those located at {(0,1), (1,0), (1,2)} belong to the White group. The corresponding vertex gadgets in the game board are shadowed with gray color.

Correctness:

We now argue that the constructed position will mimic a generalized geography game if both of the players play "correctly". If a player does not play correctly, then it leads to a losing game within a few moves. Let us consider the case that the starting vertex is of in-degree 1 and out-degree 1 as shown in Figure 10. The arguments for the other five cases are similar.

Proposition 1 *Consider Figure 4(b) and Figure 10. In the first move, if P_1 does not place stones at one of $c_1, c_2, \cdots, c_p$ points in Figure 10 and one of $a_1, a_2, \cdots, a_p$ in Figure 4(b) in each of the auxiliary zones, then P_2 can win immediately.*

Proof. Since P_1 has no threat, he cannot win in the first move. W.L.O.G., assume that P_1 does not place stone at any of $a_1, a_2, \cdots, a_p$ in Figure 4(b), then P_2 can place p white stones at $a_1, a_2, \cdots, a_p$ and win in the second move. □

Proposition 2 *P_1 will have p threats against the second move if and only if he places stones at c_1 in Figure 10 and a_1 in Figure 4(b) in each of the auxiliary zones in the first move. Moreover, P_2 has no threat before the second move.*

Proof. Clear. □

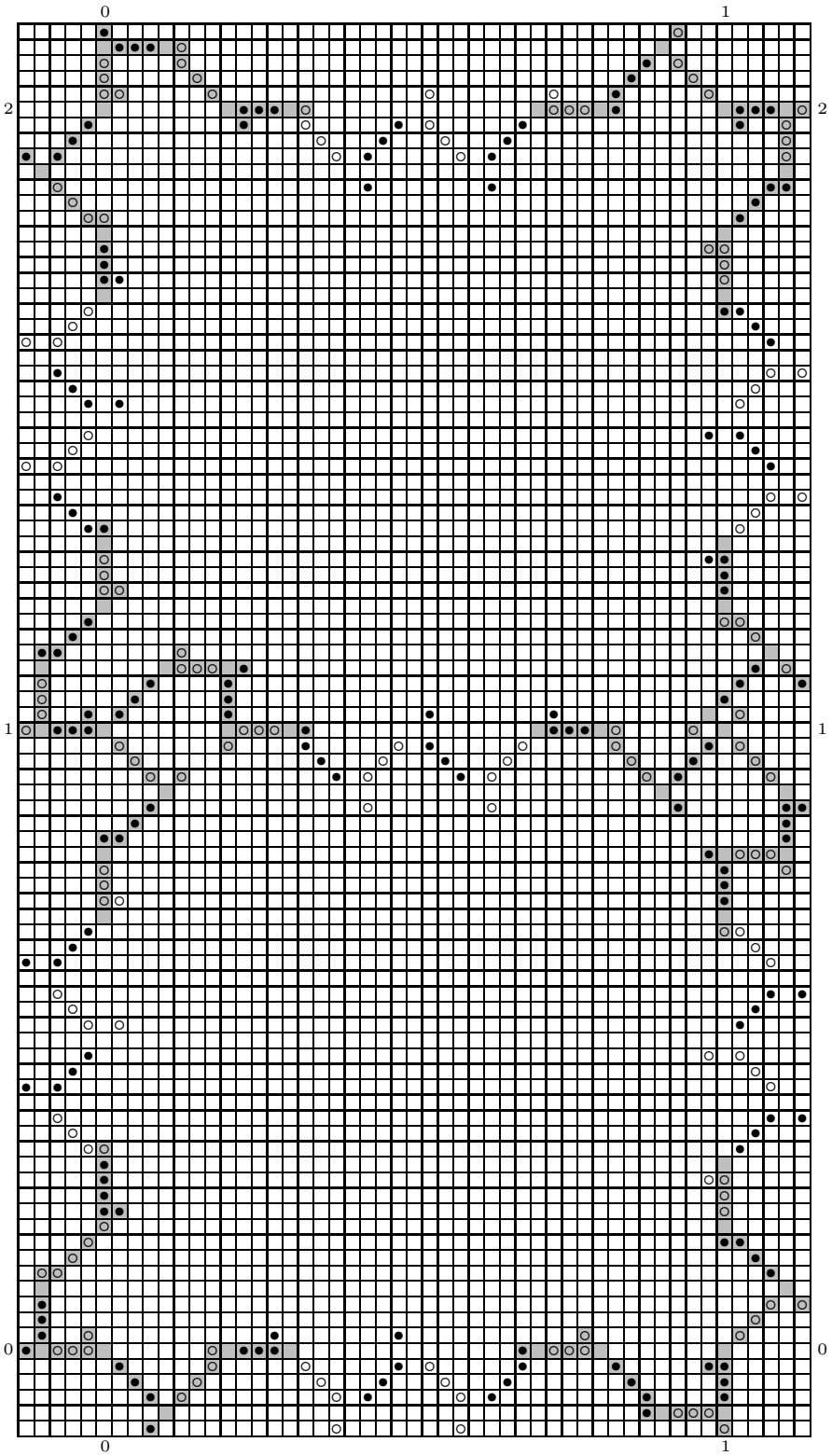
Proposition 3 *P_2 will have p threats against the third move if and only if he places stones at d in Figure 10 and w in Figure 4(b) in each of the auxiliary zones in the second move. Moreover, P_1 has no threat before the third move.*

Proof. The same as for Proposition 2. □

Proposition 4 *In the first move, if P_1 does not place a black stone at c_1 in Figure 10 or a_1 in Figure 4(b) in one of the auxiliary zones, then P_2 can win in two moves.*

Proof. By Proposition 2, P_1 will have less than p threats in the second move. W.L.O.G., we can assume that P_1 has no threat in the simulation zone (see Figure 10). Then by Proposition 3, P_2 can place $p-1$ stones on w's in the auxiliary zones to get $p-1$ threats and make P_1 have no threat. Moreover, P_2 can place one white stone at a in the winning zone in Figure 4(a) to get 2 threats. Since P_1 has no threat and P_2 has more than p threats against the third move, P_2 can win in the fourth move. □

Proposition 5 *In the second move, if P_2 does not place a white stone at d in Figure 10 or w in Figure 4(b) in one of the auxiliary zones, then P_1 can win within two moves.*



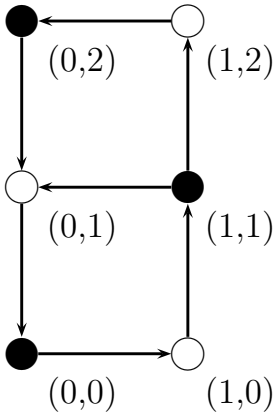


Figure 11: An example with $k - p = 4$ and the starting vertex gadget is located at (0,0).

Proof. Similar to Proposition 4. □

The arguments for the following moves are similar and can be verified easily. Next, we show the cases of different situations.

Proposition 6 *The constructed position in Figure 8(a) simulates a vertex of in-degree 1 and out-degree 2 in the Black group.*

Proof. According to the construction, when entering such a vertex, P_1 is forced to place a black stone at a, otherwise, with a similar argument to the proof of Proposition 1, P_1 would lose. Then P_2 is forced to respond a white stone at b, P_1 is forced to respond a black at c, and then P_2 is forced to respond a white stone at d. Now, P_1 can respond at e or f since $p > 1$ (if $p = 1$, P_1 is forced to respond at e). Actually, the choice of e and f simulates a vertex with out-degree 2. The arguments for the following moves are straightforward. □

Proposition 7 *The constructed position in Figure 8(b) simulates a vertex of in-degree 1 and out-degree 2 in the Black group.*

Proof. Similar to Proposition 6. □

Proposition 8 *If a player chooses to visit a visited vertex (not the starting vertex), then it leads to a losing game.*

Proof. W.L.O.G., assume P_2 revisits a vertex. Then this vertex must be of in-degree 2. Consider Figure 7(a). Assume this vertex has been visited via the left entry point and hence there must be black stones at a, c and i, and white stones at b and d, otherwise P_1 would have lost the game earlier. Next, according to the construction of the connect position, when reentering such a vertex, P_1 is forced to place one black stone at e, P_2 is forced to respond a white stone at f, and P_1 is forced to respond a black stone at g. Now, in the whole game board, P_2 has no threat and P_1 has p threats by Proposition 2. In the auxiliary zones, P_2 can get $p - 1$ threats in the following move and block P_1's threat there. However, in the simulation zone (Figure 7(a)), P_2 can only block P_1's threat but cannot create new threat at the same time, since there is already a black stone at i. Finally, similar to the argument in Proposition 4, P_2 will lose in two moves and P_1 will win. The case of Figure 7(b) is similar. □

Proposition 9 *If P_2 chooses to visit the starting vertex, then he will lose.*

Proof. Similar to Proposition 8. □

Since the play in the simulation zone will terminate when a player chooses to revisit a vertex and the opponent will then win, we have shown that the ∃-player has a winning strategy in the generalized geography game if and only if P_1 has a winning strategy from the constructed position of the $Connect(m, n, k, p, q)$ game. The setting $k - p \geq p$ and the

large enough constant distance between any two mapped coordinates ensure the required empty squares, e.g. $c_1, \cdots, c_p$ in Figure 10, will not be affected by any components of the constructed position. Note that we don't try to optimize the setting but reserve enough space, since $k-p$ can be much smaller than p and cause interference among moves if we don't require $k-p \geq p$. The reason why $k-p \geq 3$ is trivial, i.e., in Figure 7(a), there are two consecutive black stones next to b, and hence $k-p$ must be greater than 2.

Finally, we have to make sure that the reduction can be done in polynomial time. We estimate the required size for each zone of the constructed position in the following:

1. **Winning zone:** $O(k-p) \times O(k-p)$, see Figure 4(a).
2. **Simulation zone:** Since k and p are fixed constants, the size of the simulation zone is bounded by $O(V) \times O(V)$, referring to Theorem 2.
3. **Auxiliary zone:** We now determine the number of the repeated parts in Figure 4(b). Since we require that the play in the simulation zone always terminates a few moves earlier than the play in the auxiliary zones, the number of the repeated parts is related to the size of the simulation zone. Hence, the required size is $O(1) \times O(V)$.

Therefore, we obtain that $m = O(V)$ and $n = O(V)$. Since the construction mentioned above can be done in polynomial time, we have the following lemma.

Lemma 6 *The decision Connect(m, n, k, p, q) problem is PSPACE-hard when $k - p \geq \max\{3, p\}$ and $p \geq 2$.*

Theorem 4 *The decision Connect(m, n, k, p, q) problem is PSPACE-complete when $k-p \geq \max\{3, p\}$ and $p \geq 2$.*

Proof. Immediate from Lemma 4 and 6. □

Corollary 3 *To determine whether P_1 has a winning strategy in a given non-empty* **Connect6** *game position is PSPACE-complete.*

Proof. Immediate from Theorem 4. □

Corollary 4 *The decision Connect(m, n, k, p, q) problem is PSPACE-complete when $k-p \geq \max\{3, p\}$.*

Proof. Immediate from Lemma 5 and Theorem 4. □

4 Conclusion and remarks

The main results in this paper are: (1) Fairness issue: no one can win *Connect*(m, n, k, p, q) for any m, n when $q \leq p$ and $k \geq 4p+7$. (2) Complexity issue: The decision *Connect*(m, n, k, p, q) problem is PSPACE-complete when $k - p \geq \max\{3, p\}$.

Open problems: (1) Can we have a better bound than the first result, since Zetters[16] showed that P_2 can tie the game when $k \geq 8$ for $p = q = 1$? (2) Is the decision *Connect*(m, n, k, p, q) problem still PSPACE-complete if we relax the restriction on k and p?

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