3.3 Regular Grammars

Right- and Left-Linear Grammars

Definition 3.3

A grammar $G = (V, T, S, P)$ is said to be **right-linear** if all productions are of the form

$$A \to xB,$$
$$A \to x,$$

where $A, B \in V$, and $x \in T^*$. A grammar is said to be **left-linear** if all productions are of the form

$$A \to Bx,$$

or

$$A \to x.$$

A **regular grammar** is one that is either right-linear or left-linear.

Example 3.13 The grammar $G_1 = (\{S\}, \{a, b\}, S, P_1)$, with P_1 given as

$$S \to abS|a$$

is right-linear. The grammar $G_2 = (\{S, S_1, S_2\}, \{a, b\}, S, P_2)$, with productions

$$\begin{aligned} S &\to S_1ab, \\ S_1 &\to S_1ab|S_2, \\ S_2 &\to a, \end{aligned}$$

is left-linear. Both G_1 and G_2 are regular grammars.

The sequence

$$S \Rightarrow abS \Rightarrow ababS \Rightarrow ababa$$

is a derivation with G_1. From this single instance it is easy to conjecture that $L(G_1)$ is the language denoted by the regular expression $r = (ab)^* a$. In a similar way, we can see that $L(G_2)$ is the regular language $L(aab(ab)^*)$.

Example 3.14 The grammar $G = (\{S, A, B\}, \{a, b\}, S, P)$ with productions

$$
\begin{aligned}
S &\rightarrow A, \\
A &\rightarrow aB|\lambda, \\
B &\rightarrow Ab,
\end{aligned}
$$

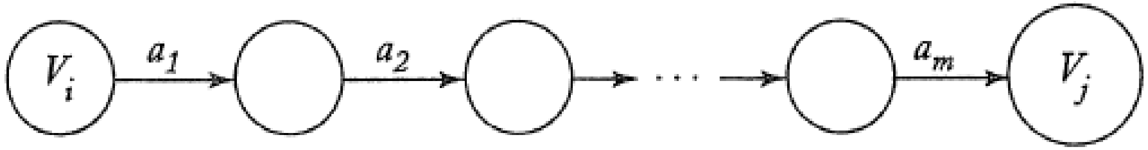
is not regular. Although every production is either in right-linear or left-linear form, the grammar itself is neither right-linear nor left-linear, and therefore is not regular. The grammar is an example of a **linear grammar**.

A linear grammar is a grammar in which at most one variable can occur on the right side of any production, without restriction on the position of this variable. Clearly, a regular grammar is always linear, but not all linear grammars are regular.

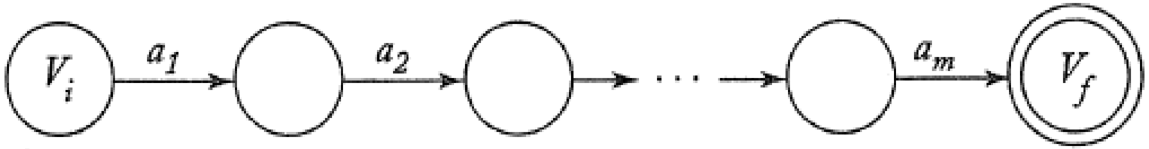
Right-Linear Grammars Generate Regular Languages

Theorem 3.3 Let $G = (V, T, S, P)$ be a right-linear grammar. Then $L(G)$ is a regular language.

Figure 3.16



Represents $V_i \to a_1 a_2 \ldots a_m V_j$



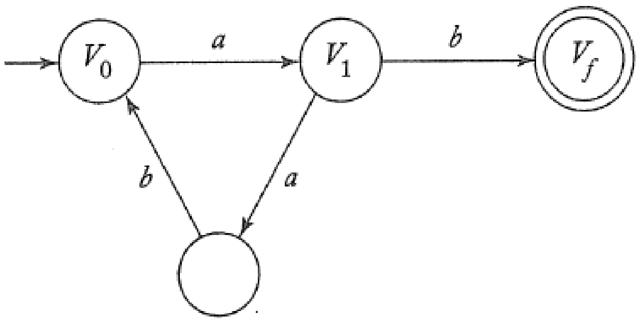
Represents $V_i \to a_1 a_2 \ldots a_m$

Example 3.15 Construct a finite automaton that accepts the language generated by the grammar

$$V_0 \to aV_1,$$
$$V_1 \to abV_0|b,$$

where V_0 is the start variable. We start the transition graph with vertices V_0, V_1, and V_f. The first production rule creates an edge labeled a between V_0 and V_1. For the second rule, we need to introduce an additional vertex so that there is a path labeled ab between V_1 and V_0. Finally, we need to add an edge labeled b between V_1 and V_f, giving the automaton shown in Figure 3.17. The language generated by the grammar and accepted by the automaton is the regular language $L\left((aab)^* ab\right)$.

Figure 3.17



Right-Linear Grammars for Regular Languages

Theorem 3.4 If L is a regular language on the alphabet Σ, then there exists a right-linear grammar $G = (V, \Sigma, S, P)$ such that $L = L(G)$.

Proof: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a dfa that accepts L. We assume that $Q = \{q_0, q_1, ..., q_n\}$ and $\Sigma = \{a_1, a_2, ..., a_m\}$. Construct the right-linear grammar $G = (V, \Sigma, S, P)$ with

$$V = \{q_0, q_1, ..., q_n\}$$

and $S = q_0$. For each transition

$$\delta(q_i, a_j) = q_k$$

of M, we put in P the production

$$q_i \to a_j q_k. \tag{3.5}$$

In addition, if q_k is in F, we add to P the production

$$q_k \to \lambda. \tag{3.6}$$

We first show that G defined in this way can generate every string in L. Consider $w \in L$ of the form

$$w = a_i a_j \cdots a_k a_l.$$

For M to accept this string it must make moves via

$$\begin{aligned} \delta(q_0, a_i) &= q_p, \\ \delta(q_p, a_j) &= q_r, \\ &\vdots \\ \delta(q_s, a_k) &= q_t, \\ \delta(q_t, a_l) &= q_f \in F. \end{aligned}$$

By construction, the grammar will have one production for each of these δ's. Therefore, we can make the derivation

$$\begin{aligned} q_0 \Rightarrow a_i q_p \Rightarrow a_i a_j q_r &\stackrel{*}{\Rightarrow} a_i a_j \cdots a_k q_t \\ &\Rightarrow a_i a_j \cdots a_k a_l q_f \Rightarrow a_i a_j \cdots a_k a_l, \end{aligned} \tag{3.7}$$

with the grammar G, and $w \in L(G)$.

Conversely, if $w \in L(G)$, then its derivation must have the form (3.7). But this implies that

$$\delta^* (q_0, a_i a_j \cdots a_k a_l) = q_f,$$

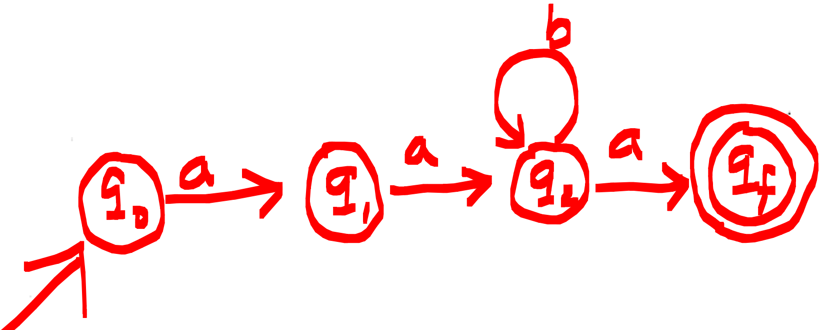
completing the proof. ■

Example 3.16 Construct a right-linear grammar for $L(aab^*a)$. The transition function for an nfa, together with the corresponding grammar productions, is given in Figure 3.18. The result was obtained by simply following the construction in Theorem 3.4. The string $aaba$ can be derived with the constructed grammar by

$$q_0 \Rightarrow aq_1 \Rightarrow aaq_2 \Rightarrow aabq_2 \Rightarrow aabaq_f \Rightarrow aaba.$$

Figure 3.18

nfa	productions
$\delta(q_0, a) = \{q_1\}$	$q_0 \rightarrow aq_1$
$\delta(q_1, a) = \{q_2\}$	$q_1 \rightarrow aq_2$
$\delta(q_2, b) = \{q_2\}$	$q_2 \rightarrow bq_2$
$\delta(q_2, a) = \{q_f\}$	$q_2 \rightarrow aq_f$
$q_f \in F$	$q_f \rightarrow \lambda$



Equivalence of Regular Languages and Regular Grammars

Theorem 3.5 A language L is regular if and only if there exists a left-linear grammar G such that $L = L(G)$.

Proof: We only outline the main idea. Given any left-linear grammar with productions of the form

$$A \rightarrow Bv,$$

or

$$A \rightarrow v,$$

we construct from it a right-linear grammar $\widehat{G}$ by replacing every such production of G with

$$A \rightarrow v^R B,$$

or

$$A \rightarrow v^R,$$

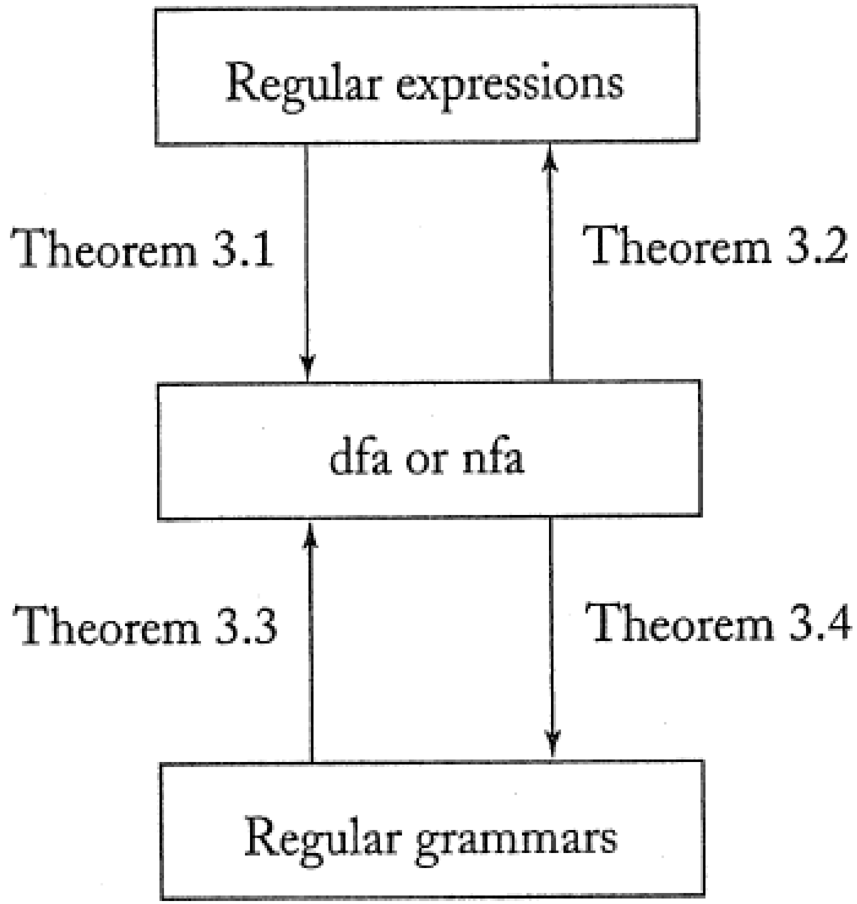
respectively. A few examples will make it clear quickly that $L(G) = \left(L\left(\widehat{G}\right)\right)^R$. Next, we use Exercise 12, Section 2.3, which tells us that the reverse of any regular language is also regular. Since $\widehat{G}$ is right-linear, $L\left(\widehat{G}\right)$ is regular. But then so are $L\left(\left(\widehat{G}\right)\right)^R$ and $L(G)$. ■

Putting Theorems 3.4 and 3.5 together, we arrive at the equivalence of regular languages and regular grammars.

Theorem 3.6

A language L is regular if and only if there exists a regular grammar G such that $L = L(G)$.

Figure 3.19



Regular expressions
Theorem 3.1
Theorem 3.2
dfa or nfa
Theorem 3.3
Theorem 3.4
Regular grammars