Problem 1 (20%). Prove that, for any vector $v \in \mathbb{R}^n$,

$$\frac{|v|_1}{\sqrt{n}} \leq \|v\|_2 \leq |v|_1,$$

where $|v|_1 := \sum_i |v_i|$ is the L_1-norm and $\|v\|_2 := \left(\sum_i v_i^2\right)^{1/2}$ is the L_2-norm of v.

Hint: Use the Cauchy-Schwarz inequality, i.e., $|u \cdot v| \leq \|u\|_2 \|v\|_2$ for any $u, v \in \mathbb{R}^n$.

Problem 2 (20%). Let A be a square symmetric matrix and λ be an eigenvalue of A. Prove that, for any $k \in \mathbb{N}$, λ^k is an eigenvalue of A^k.

Problem 3 (20%). Let G be an n-vertex d-regular bipartite graph and A be the normalized adjacency matrix of G. Prove that, there exists a vector $v \in \mathbb{R}^n$ such that

$$Av = -v.$$

Generalize the construction to non-regular bipartite graphs, i.e., for any bipartite graph G' with column-normalized adjacency matrix A', prove that A' has an eigenvalue -1.

Note: A' is also called the *random-walk* matrix of G'.

Problem 4 (20%). Let $G = (V, E)$ be a d-regular graph and P be a random walk of length t in G. Prove that, for any edge $e \in E$ and any $1 \leq i \leq t$,

$$\Pr\left[\ e \text{ is the } i^{th}\text{-edge of } P\ \right] = \frac{1}{|E|}.$$

Hint: Prove by induction on i.

Problem 5 (20%). Let $G = (V, E)$ be an (n, d, λ)-expander and $S \subseteq V$ be a vertex subset. Prove that,

$$\Pr_{(u,v)\in E}\left[\ u, v \in S\ \right] \leq \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right),$$

i.e., for any $(u, v) \in E$, the probability that both u, v are in S is bounded by $\frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$.

Hint: Use the fact that $|E(S, S)| = (d|S| - |E(S, T)|)/2$. Apply the crossing lemma.