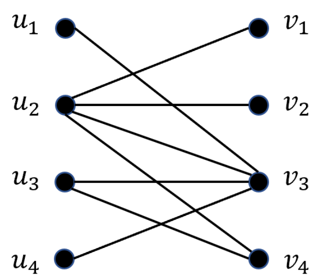
Problem 1 (20%). Consider the following graph. Identify a maximum-size matching and a minimum-size vertex cover for it.



Problem 2 (20%). Let G be a bipartite graph with partite sets A and B, and M, M' be two matchings. Suppose that, M matches the vertices in $S \subseteq A$ and M' matches the vertices in $T \subseteq B$. Prove that there is a matching that matches all the vertices in $S \cup T$.

Hint: Consider $M \cup M'$.

Problem 3 (20%). Let G be a bipartite graph with partite sets X and Y. Prove that G has a matching of size t if and only if for all $A \subseteq X$,

$$|N(A)| \;\geq\; |A| + t - |X| \;=\; t - |X - A|.$$

Hint: Add $|X| - t$ new vertices to Y and connect these vertices to every vertex in X.

Problem 4 (20%). Let G be a bipartite graph with partite sets X and Y. Define

$$\delta(G) := \max_{A \subseteq X} \left(|A| - |N(A)|\right),$$

i.e., $\delta(G)$ measures the worst violation of the Hall's matching condition. Note that, $\delta(G) \geq 0$ since $A = \emptyset$ is considered as a subset of X. Use the statement in Problem 3 to prove that, G has a maximum matching of size $|X| - \delta(G)$.

Problem 5 (20%). Let G be a bipartite graph with partite sets X and Y. Assume the same notation $\delta(G)$ as Problem 4. Show that, the largest independent set of G has size $|Y| + \delta(G)$.