Problem 1 (20%). Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n+1$.

For each n, exhibit two sets of size n to show that the above results are the best possible, i.e., sets of size $n+1$ are necessary.

Hint: Use pigeonholes $(2i, 2i-1)$ and $(i, 2n-i+1)$ for $1 \leq i \leq n$.

Problem 2 (20%). Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in V such that, no adjacent vertices are colored the same. Prove that, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

Hint: Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

Problem 3 (20%). Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set of G. Prove the following dual version of Turán's theorem:

> If G is a graph with n vertices and $nk/2$ edges, where $k \geq 1$, then we have
>
> $$\alpha(G) \geq n/(k+1).$$

Problem 4 (20%). Consider the following two problems regarding Markov's and Chebyshev's inequalities.

- For any positive integer k, describe a non-negative random variable X such that

  $$\Pr[\, X \geq k \cdot \mathrm{E}[X] \,] \;=\; \frac{1}{k}.$$

  Note that, this shows that Markov's inequality is as tight as it could possibly be.

- Can you provide an example that shows that Chebyshev's inequality is tight? If not, explain why not.

Problem 5 (20%). Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these random bits in any order. Let Y_i be the exclusive-or (XOR) of the i^{th} pair of bits, and let $Y := \sum_{1 \leq i \leq m} Y_i$.

- Show that $Y_i = 0$ and $Y_i = 1$ with probability $1/2$ each.
- Show that $\mathrm{E}[\, Y_i \cdot Y_j \,] = \mathrm{E}[Y_i] \cdot \mathrm{E}[Y_j]$ for any $1 \leq i, j \leq m$ and derive $\mathrm{Var}[Y]$.
- Use Chebyshev's inequality to derive a bound on $\Pr[\, |Y - \mathrm{E}[Y]| \geq n \,]$.