Problem 1 (20%). Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

Problem 2 (20%). Prove that for any two sets I, J with $I \subseteq J$, we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

Hint: Rewrite the summation and apply the binomial theorem (in slides # 1a).

Problem 3 (20%). Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k elements and each element belongs to k sets. Let $k \geq 10$. Show that there exists at least one valid 2-coloring of the elements.

Hint: Define proper events for the sets and apply the symmetric version of the local lemma.

Problem 4 (20%). We proved the asymmetric version of the local lemma in lecture #4. Assume that the statement of this lemma holds. Furthermore, assume that

1. $\Pr[A_i] \leq p$ for all i, and
2. $ep(d+1) \leq 1$.

Prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$, i.e., use Theorem 19.2 to prove the statement of Theorem 19.1.

Hint: Let $x(A_i) = \frac{1}{d+1}$ for all $1 \leq i \leq n$. Use the inequality $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$ obtained by the limit formula of $1/e$ and the fact that it converges from the above.

Problem 5 (20%). Let X be a finite set and $A_1, A_2, \ldots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \leq i \leq m$, we say that the block B_i is λ-*large* if

$$\frac{|B_i|}{|A_i|} \geq \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1 - \lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.