Combinatorial Mathematics

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Monday 18:30 – 21:20

Outline

- The Pigeonhole principle
  - The Erdős-Szekeres Theorem
  - The Dilworth Lemma for Posets
  - Mantel’s Theorem
  - Turán’s Theorem

The Pigeonhole Principle

(aka Dirichlet's principle)

If a set of size at least r is partitioned into s sets, then some class receives at least $\lceil r/s \rceil$ elements.

Proposition 1.

In any graph, there exist two vertices with the same degree.

- Let $G = (V, E)$ be a graph with $|V| = n$.
- The degree of any vertex is between 0 and $n - 1$.
  - If there is a vertex with degree 0, then there exists no vertex with degree $n - 1$, and vice versa.
  - Hence, there are at most $n - 1$ different values for the vertex degrees, while there are n vertices.
  - By the pigeonhole principle,
    at least two vertices have the same degree.

Independent Set & Chromatic Number

- Let $G = (V, E)$ be a graph.
- Let
  - $\alpha(G)$ be the maximum size of any independent set for G.
  - $\chi(G)$ be the chromatic number of G,

    i.e., the minimum number of colors required to color V such that,

    no adjacent vertices are colored the same.

Independent Set & Chromatic Number

- Let $G = (V, E)$ be a graph.
  - Let $\alpha(G)$ denote the size of maximum independent set for G.
  - Let $\chi(G)$ denote the chromatic number of G.

- Consider a coloring of V that uses $\chi(G)$ colors.
  - Let $V_1, V_2, \ldots, V_{\chi(G)}$ be the partition of the vertices by their colors.
    - For any $1 \leq i \leq \chi(G)$,

      the set V_i is an independent set for G.

Proposition 2.

In any graph G with n vertices, $n \leq \alpha(G) \cdot \chi(G)$.

- Proof 1.
  - Consider a coloring of V that uses $\chi(G)$ colors and $V_1, V_2, \ldots, V_{\chi(G)}$ be the partition of the vertices by their colors.
  - Since V_i is an independent set, $|V_i| \leq \alpha(G)$.
  - Hence,

$$n = \sum_{1 \leq i \leq \chi(G)} |V_i| \leq \alpha(G) \cdot \chi(G).$$

Proposition 2.

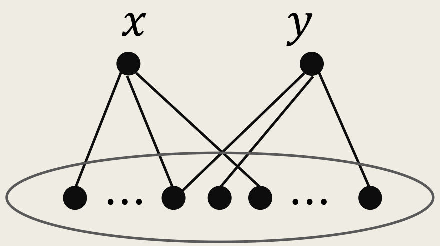
In any graph G with n vertices, $n \leq \alpha(G) \cdot \chi(G)$.

- Proof 2.
  - Consider a coloring of V that uses $\chi(G)$ colors and let $V_1, V_2, \ldots, V_{\chi(G)}$ be the partition of the vertices by their colors.
  - By the pigeonhole principle, there exists some i with $|\mathrm{V}_i| \geq \frac{n}{\chi(G)}$.
  - Since V_i is an independent set, $\alpha(G) \geq |V_i|$.
  - By the above two inequalities, $n \leq \alpha(G) \cdot \chi(G)$.

<u>Proposition 3.</u>

Let G be a graph with n vertices. If every vertex has a degree of at least $(n-1)/2$, then G is connected.

- Proof.
  - We prove that, for any pair of vertices, say, x and y, either x and y are *<u>adjacent</u>* or *<u>have a common neighbor</u>*.

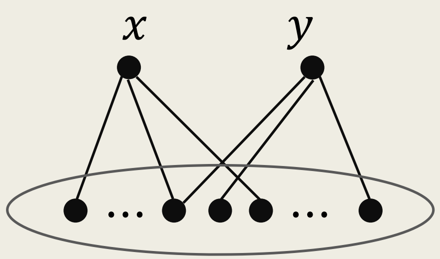


  - If x and y are not adjacent, then there are at least $n-1$ edges connecting them to the remaining vertices.
  - Since there are only $n-2$ other vertices, at least two of these $n-1$ edges connect to the same vertex.

Some Remark.

- The statement from Proposition 3 is the best possible.
  - To see that, consider the graph that consists of two disjoint complete graphs, each having $n/2$ vertices.

    Then every vertex has degree $n/2 - 1$, and the graph is disconnected.



- Also note that, we also proved that, if every vertex has degree at least $(n-1)/2$, then the diameter of the graph is at most two.

The Erdős-Szekeres Theorem

Increasing / Decreasing Sequences

- Let $A = (a_1, a_2, \dots, a_n)$ be a sequence of n distinct numbers.
  - A sequence of B with length k is called a *subsequence* of A, if the elements of B appear in the same order in which they appear in A, i.e.,

    $B = \left(a_{i_1}, a_{i_2}, \dots, a_{i_k}\right)$, where $i_1 < i_2 < \cdots < i_k$.

- A sequence is said to be increasing if $a_1 < a_2 < \cdots < a_n$ and decreasing if $a_1 > a_2 > \cdots > a_n$.

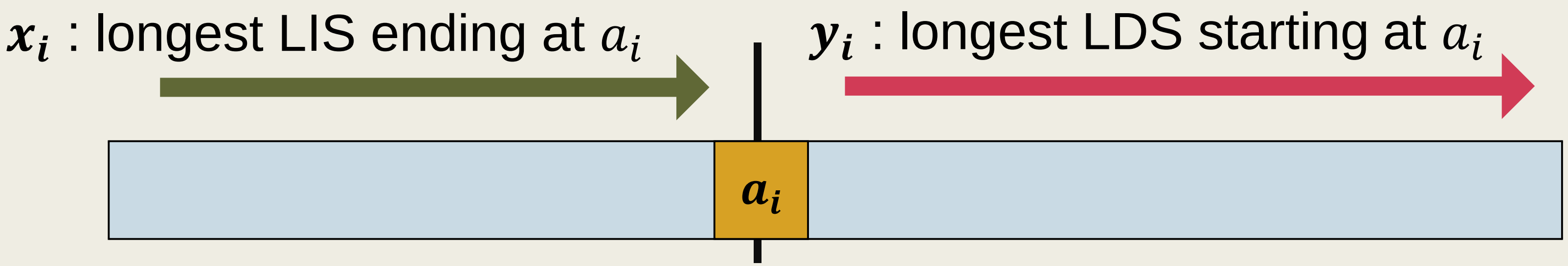
Theorem 5 (Erdős-Szekeres 1935).

Let $A = (a_1, a_2, \dots, a_n)$ be a sequence of n distinct numbers. If $n \geq sr + 1$, then A has either an increasing subsequence of length $s + 1$ or a decreasing subsequence of length $r + 1$.

- Proof. (due to Seidenberg 1959).

  For any $1 \leq i \leq n$, associate a_i with a pair (x_i, y_i), where

  - x_i is the length of the *longest **increasing** subsequence* ***ending at*** a_i.
  - y_i is the length of the *longest **decreasing** subsequence* ***starting at*** a_i.



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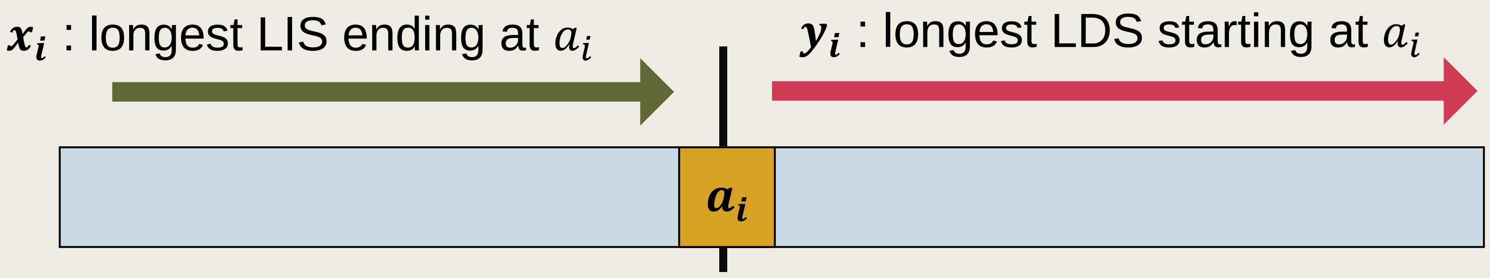
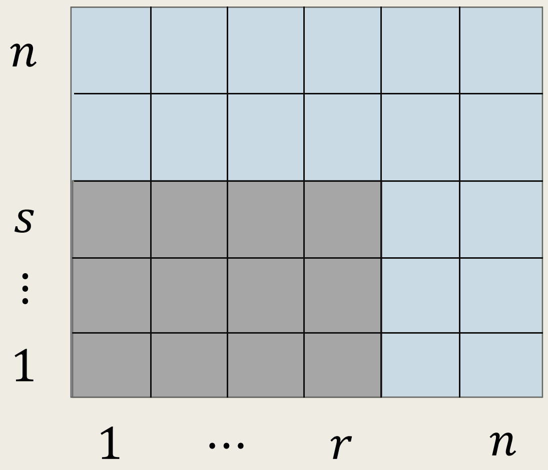


- For $1 \leq i < j \leq n$, we have $(x_i, y_i) \neq (x_j, y_j)$.
  - If $a_i < a_j$, then $x_j \geq x_i + 1$.
  - If $a_i > a_j$, then $y_i \geq y_j + 1$.

One of the two conditions must hold, since the elements are distinct.

- For any $1 \leq i < j \leq n$, we have $(x_i, y_i) \neq (x_j, y_j)$.
  - If $a_i < a_j$, then $x_j \geq x_i + 1$.
  - If $a_i > a_j$, then $y_i \geq y_j + 1$.



- Consider the $n \times n$ grids.
  - By the above observation, all the elements a_i correspond to a distinct grid.

- Consider the $s \times r$ submatrix.
  - Since $n > s \cdot r$, for some i, the element a_i corresponds to some grid outside the $s \times r$ submatrix.
  - Hence, either $x_i > s$ or $y_i > r$.

The Dilworth Lemma

for Partially Ordered Sets (Posets)

Partial Order.

- A *partial order* on a set P is a *binary relation* $\preccurlyeq$ that is
  - *(reflexive).* $a \preccurlyeq a$, for all $a \in P$,
  - *(antisymmetric).* If $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a = b$.
  - *(transitive).* If $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.

- Two elements $a, b \in P$ are said to be *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Chain and Antichain.

- Let P be a set with partial order $\preccurlyeq$.
  - A subset $C \subseteq P$ is called a *chain*,

    if every pair of elements in C is comparable.

  - Dually, a subset $C \subseteq P$ is called an *antichain*,

    if none of the pairs in C is comparable.

Chain and Antichain.

- For example,

  let $P = \{ 1, 2, 3, 4, 5, a, b, c, d \}$ and define the partial order $\leq$ as

$$1 \leq 2 \leq 3 \leq 4 \leq 5, \text{ and}$$

$$a \leq b \leq c \leq d.$$

  - Then, $\{4,2,3\}$ and $\{c, d\}$ are two chains,

    and $\{2, c\}$ is an antichain.

Lemma 6 (Dilworth 1950).

Let P be a set with a partial order $\preccurlyeq$.
If $|P| \geq sr + 1$, then there exists either a chain of size $s + 1$ or an antichain of size $r + 1$.

- Proof.
  - For any $a \in P$,
    let $\ell(a)$ denote the ***length*** of the ***longest chain ending at*** $\boldsymbol{a}$.
  - Suppose that there exists no chain of size $s + 1$.
    - Then $\ell(a) \leq s$ for all $a \in P$.
    - We will show that, there exists an antichain of size $r + 1$.

- For any $a \in P$,
  let $\ell(a)$ denote the length of the longest chain ending at a.

- For $1 \leq i \leq s$, let A_i be the set of elements a with $\ell(a) = i$.

  - Then, $\boldsymbol{A_i}$ **must be an antichain**, for all $1 \leq i \leq s$.

    - Consider any $a, b \in A_i$ with $a \neq b$.
      By assumption, we have $\ell(a) = \ell(b)$.

    - If a and b are comparable, say, $a \preccurlyeq b$,
      then, we add b to the longest chain ending at a.

      This gives a chain ending at b with length $\ell(b) + 1$,
      a contradiction.

- Suppose that there exists no chain of size $s+1$.
  - Then $\ell(a) \leq s$ for all $a \in P$.

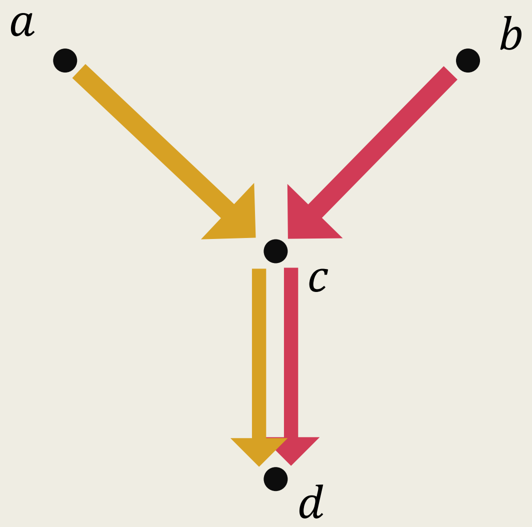
- For $1 \leq i \leq s$, let A_i be the set of elements a with $\ell(a) = i$.
  - Then, $\boldsymbol{A_i}$ **is an antichain**, for all $1 \leq i \leq s$.
  - $A_i \cap A_j = \emptyset$ for all $i \neq j$.
  - $A_1, A_2, \ldots, A_s$ forms a partition of P.

- Since $|P| \geq sr+1$,

  by the pigeonhole principle, $|A_i| \geq r+1$ for some i.

Some Note.

- The proof given in the textbook is wrong.
  - The greatest elements chosen in different maximal chains can be identical, and hence, comparable.



For example,

the two maximal chains, $\{a, c, d\}$ and $\{b, c, d\}$,

share the same greatest element d.

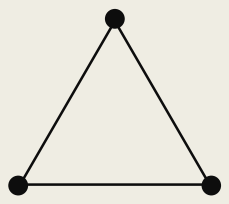
The Mantel's Theorem

How many edges can a *triangle-free graph* have?

Alternatively,
how many edges can we add to a graph without creating a triangle?

The Maximum Number of Edges in a Triangle-free Graph.

- A triangle is a complete graph of 3 vertices.



- We know that, bipartite graphs do not contain any triangle.
  - So, $n^2/4$ edges are possible,

    achieved by complete bipartite graphs with two $n/2$ partite sets.
  - It turns out that, $n^2/4$ is also the best possible.

Theorem 7 (Mantel 1907).

If an n-vertex graph has more than $n^2/4$ edges, then it contains a triangle.

- Proof 1.
  - Let $G = (V, E)$ with $|V| = n$ and $|E| = m$.
  - Assume that G has no triangles.
    - Consider any $e = (x, y) \in E$.

      The pigeonhole principle guarantees that

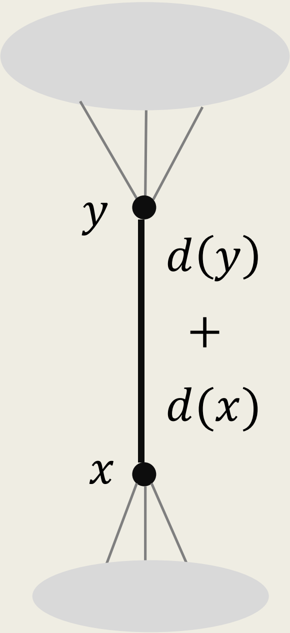
$$d(x) + d(y) \leq n\,.$$

$|V| = n$. If $d(x) + d(y) > n$, x and y must share a common neighbor and they form a triangle.

- Proof 1.
  - Let $G = (V, E)$ with $|V| = n$ and $|E| = m > n^2/4$.
  - Assume that G has no triangles. Consider any $e = (x, y) \in E$.

    The pigeonhole principle guarantees that

$$d(x) + d(y) \leq n\,.$$



- Summing over all the edges, we obtain

$$\sum_{x \in V} d(x)^2 \quad = \boxed{\sum_{(x,y) \in E} \big(d(x) + d(y)\big) \;\leq\; mn\,.}$$

By the double counting principle.

– We obtain $$\boxed{\sum_{x\in V} d(x)^2 \;\le\; mn\,.}$$

For any vector $u, v \in \mathbb{R}^n$,
$$|u \cdot v| \le \|u\| \cdot \|v\|\,.$$

– Apply the ***<u>Cauchy-Schwarz inequality</u>*** to lower-bound $\sum_{x\in V} d(x)^2$.

Define two vectors $\begin{cases} u = (\,1,\ 1,\ \ldots\ ,1\,) \\ v = \big(d(v_1), d(v_2), \ldots, d(v_n)\big) \end{cases}$.

We have $$\boxed{|V| \cdot \sum_{x\in V} d(x)^2 \;\ge\; \left(\sum_{x\in V} d(x)\right)^2} \;=\; 4m^2\,.$$

Hence, $m \le n^2/4$.

$\sum_{x\in V} d(x) = 2m$ by the double counting principle.

Theorem 7 (Mantel 1907).

If an n-vertex graph has more than $n^2/4$ edges, then it contains a triangle.

- Proof 2.
  - In the second proof, we count the number of edges using the property of the ***maximum independent sets***.
  - Let $G = (V, E)$ with $|V| = n$.

    Assume that G has no triangles.
    - We will show that $|E| \leq n^2/4$.

(∗) If not, we get a triangle.

- Assume that G has no triangles.
- For any $v \in V$, ***the neighbors*** of v ***form an independent set***.
- Let $A \subseteq V$ be a maximum independent set (MIS) in G.
  - None of vertex pairs in A is connected by an edge.
  - Hence, ***every edge in G*** connects some vertex in $B \coloneqq V \setminus A$.
  - We obtain

$$|E| \leq \boxed{\sum_{x \in B} d(x) \leq \sum_{x \in B} |A|} = \boxed{|A| \cdot |B| \leq \left(\frac{|A| + |B|}{2}\right)^2} = n^2/4 .$$

By $(*)$ and A being an MIS for G.

Arithmetic and geometric mean inequality.

Turán’s Theorem

How many edges can a K_ℓ-*free graph* have?

Alternatively,
how many edges can we add to a graph without creating a clique of size ℓ ?

The Maximum Number of Edges in a K_ℓ-free Graph.

- A ℓ-clique, denoted K_ℓ, is a complete graph on ℓ vertices.

- The Mantel's theorem states that,

  any K_3-free graph has at most $n^2/4$ edges.

  - What about k-cliques with $k > 3$?

Theorem 8 (Turán 1941).

If a graph $G = (V, E)$ with n vertices contains no $(k+1)$-cliques, where $k \geq 2$, then

$$|E| \leq \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2} .$$

- Proof.
  - The case $k = 2$ is proved by the Mantel's theorem.
    Suppose that $k \geq 3$.
  - Let's prove by induction on n.
    The case with $n = 1$ is trivial. Suppose that the inequality holds for graphs with at most $n - 1$ vertices.

- The case with $n = 1$ is trivial. Suppose that the inequality holds for graphs with at most $n - 1$ vertices.

- Let $G = (V, E)$ be an n-vertex graph that has no $(k + 1)$-cliques and a maximal number of edges.

  Hence,

  - Adding any new edge to G will create a $(k + 1)$-clique.
  - G contains at least one k-clique.

    Let A be a k-clique in G, and let $B \coloneqq V \setminus A$.

  - Let $e_A, e_B, e_{A,B}$ denote the number of edges in A, in B, and that between A and B, respectively.

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  - Let A be a k-clique in G, and let $B \coloneqq V \setminus A$.
  - Let $e_A, e_B, e_{A,B}$ denote the number of edges in A, in B, and that between A and B, respectively.
  - We have $\boxed{e_A = \binom{k}{2} = k(k-1)/2.}$

    By the induction hypothesis, $\boxed{e_B \leq \left(1 - \frac{1}{k}\right) \cdot \frac{(n-k)^2}{2}.}$

    Each $v \in B$ is adjacent to at most $k - 1$ vertices in A.

    Hence, $\boxed{e_{A,B} \leq (k-1) \cdot (n-k).}$

G has no $(k+1)$-cliques

- We have $\boxed{e_A = \binom{k}{2} = k(k-1)/2.}$

$\boxed{e_B \leq \left(1 - \frac{1}{k}\right) \cdot \frac{(n-k)^2}{2}\ .}$ $\boxed{e_{A,B} \leq (k-1) \cdot (n-k).}$

- We obtain that

$$
\begin{aligned}
|E| &= e_A + e_B + e_{A,B} \\
&\leq \frac{k(k-1)}{2} + \left(1 - \frac{1}{k}\right) \cdot \frac{(n-k)^2}{2} + (k-1)(n-k) \\
&= \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2}\ .
\end{aligned}
$$