

Combinatorial Mathematics

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Monday 18:30 – 21:20

Outline

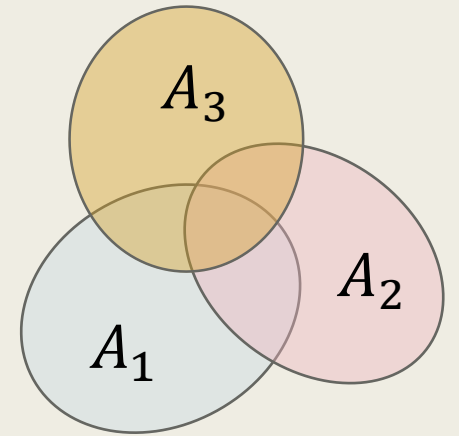
- The Lovász Local Lemma
 - Symmetric & Asymmetric versions
- Ex 1. Disjoint Cycles
- Ex 2. 2-Colorable Families

The Scenario

- To prove that $\Pr[\cap_i \overline{A_i}] > 0$ for a collection of bad events A_i , where

- A_1 : undesirable event #1
- A_2 : undesirable event #2
- ...

- $\cap_i \overline{A_i}$: the event that none of the bad events happen

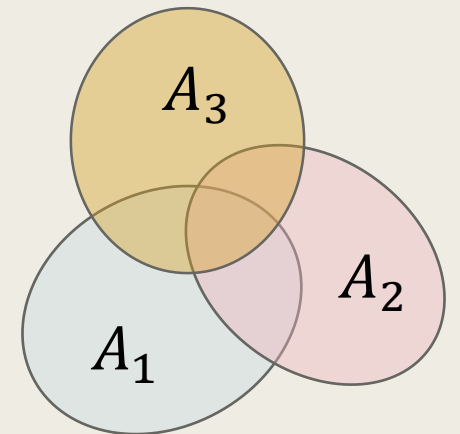


The Scenario

- To prove that $\Pr[\cap_i \overline{A_i}] > 0$ for a collection of bad events A_i
- When A_i are mutually independent and $\Pr[A_i] < 1$ for all i , then

$$\Pr\left[\cap_i \overline{A_i}\right] = \prod_i \Pr[\overline{A_i}] = \prod_i (1 - \Pr[A_i]) > 0.$$

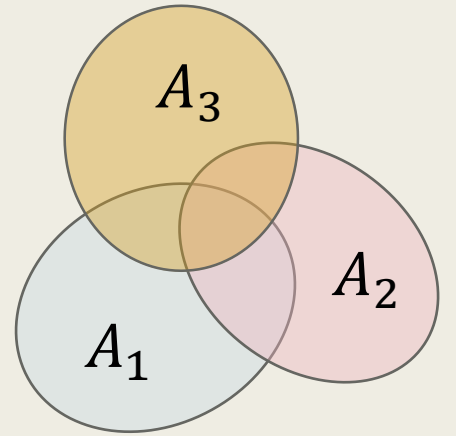
- This argument works only when A_i satisfy strong independent requirement.



- When A_i are not independent, but $\sum_i \Pr[A_i] < 1$,
we can apply union bound on A_i .

$$\Pr\left[\bigcup_i A_i\right] \leq \sum_i \Pr[A_i], \quad \text{and}$$

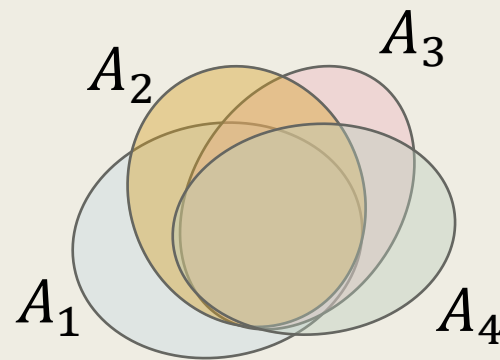
$$\begin{aligned} \Pr\left[\bigcap_i \overline{A_i}\right] &= \Pr\left[\overline{\bigcup_i A_i}\right] = 1 - \Pr\left[\bigcup_i A_i\right] \\ &\geq 1 - \sum_i \Pr[A_i] > 0. \end{aligned}$$



However, when $\sum_i \Pr[A_i] \geq 1$, the approach is inconclusive.

The Pros and the Cons

- Method 1 has the exact probability on $\Pr[\cap_i \overline{A_i}]$.
However, it works only when A_i are independent.
- Method 2 can be used with dependency.
However, union bound is loose and often it becomes inconclusive.



The shared area is counted a number of times in the union bound.

The Lovász Local Lemma (LLL)

- The Lovász Local Lemma provides a possible solution to the above scenario.

- Roughly speaking, it says that,

We need to define what they mean.

when the events are “mostly independent” and
individually “not too likely to happen”,

then there is a positive probability that ***none of the events will occur***.

A revised union bound that takes the dependency of the events into considerations.

Some Definitions

Mutual Independence

- An event A is mutually independent of the events B_1, B_2, \dots, B_k ,
if for any Boolean combination $C = \{C_1, C_2, \dots, C_k\}$ of B_1, B_2, \dots, B_k ,
where $C_i \in \{B_i, \bar{B}_i\}$ for all $1 \leq i \leq k$,
we always have

$$\Pr[A|C] = \Pr[A] .$$

Mutual Independence

- Note that, by the definition,
if A is *mutually independent* of the events B_1, B_2, \dots, B_k ,
then A is *mutually independent* of **any subsets of** B_1, B_2, \dots, B_k .

Refer to the jamboard for a sketch of the proof.

Why do we need this definition?

- Note that, it is possible that

An event A is individually independent of the events B_1, B_2, \dots, B_k , but depends on some combination of them.

- For example, suppose that a fair coin is flipped twice, and let

A : both flips are the same.

B_i : the i^{th} -flip is a head.

$$\Pr[A \mid B_i] = 1/2 = \Pr[A] .$$

Then A is independent of B_1 and B_2 separately, but

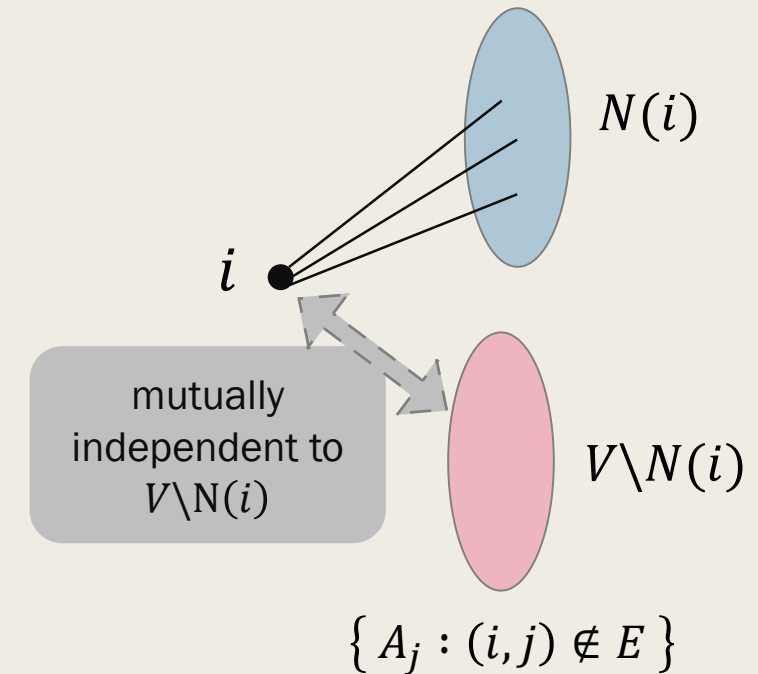
$$\Pr[A \mid B_1 B_2] = 1 \neq \Pr[A] .$$

Dependency Graph of the Events

- Let A_1, A_2, \dots, A_n be events.
 - A graph $G = (V, E)$ with $V = \{1, 2, \dots, n\}$ is said to be
a dependency graph for A_1, A_2, \dots, A_n

if for all $1 \leq i \leq n$,

A_i is *mutually independent* to $\{ A_j : (i, j) \notin E \}$.



Note that, by the definition, dependency graph is not unique.

The Lovász Local Lemma

(Symmetric version)

The Lovász Local Lemma

Theorem 19.1 (Erdős-Lovász 1975).

Let A_1, A_2, \dots, A_n be events with $\Pr[A_i] \leq p$ for all i , and ***d* be the maximum degree** of a dependency graph for the events.

If ***ep(d + 1) ≤ 1***, then

$$\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] > 0 .$$

A slightly weaker version

- The following (weaker) version is sometimes more handy to apply.

Theorem (Erdős-Lovász 1975).

Let A_1, A_2, \dots, A_n be events with $\Pr[A_i] \leq p$ for all i , and let d be the maximum degree of a dependency graph.

If $4pd \leq 1$, then $\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] > 0$.

In fact, this was the original version of LLL when it first appeared.

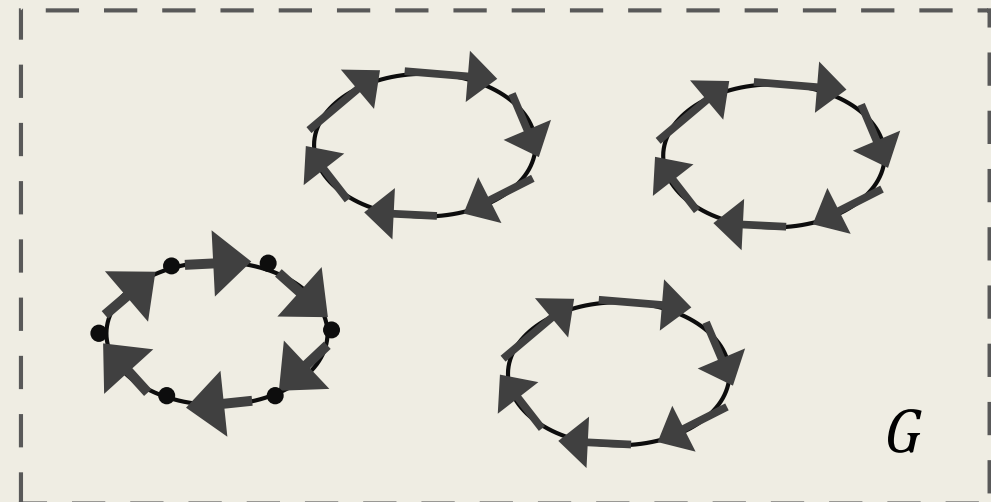
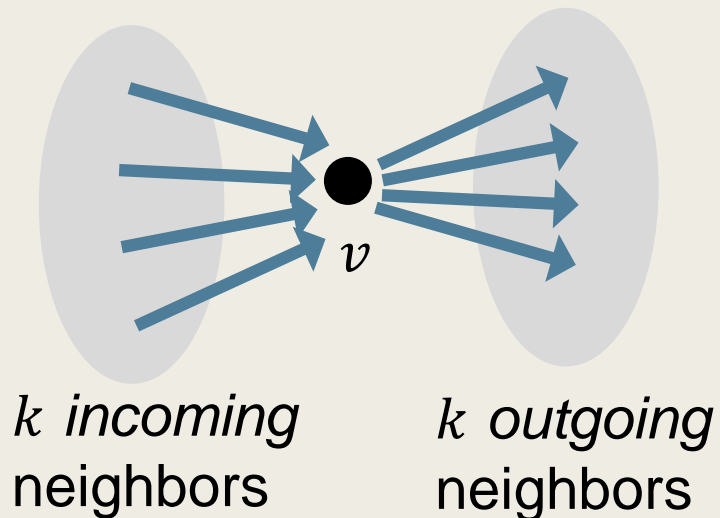
Ex1.

Disjoint Cycles

- A directed graph is said to be ***k-regular***, if the *in-degree* and the *out-degree* of every vertex are both k .

Theorem 19.4 (Erdős 1963a).

Every finite k -regular directed graph has a collection of $\lfloor k/(3 \ln k) \rfloor$ vertex disjoint cycles.



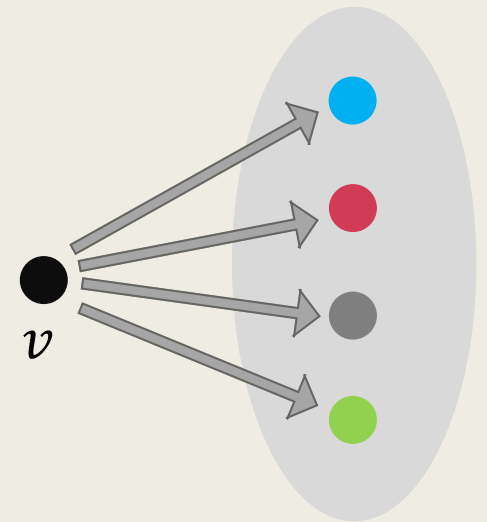
Theorem 19.4 (Erdős 1963a).

Every finite k -regular directed graph has a collection of $\lfloor k/(3 \ln k) \rfloor$ vertex disjoint cycles.

- Consider a uniform random coloring of the vertices using $r := \lfloor k/(3 \ln k) \rfloor$ colors.

To prove the lemma, we will show that,
there exists a coloring such that,

every vertex has all the r colors in its out-neighbors.



Why does this suffice?

Theorem 19.4 (Erdős 1963a).

Every finite k -regular directed graph has a collection of $\lfloor k/(3 \ln k) \rfloor$ vertex disjoint cycles.

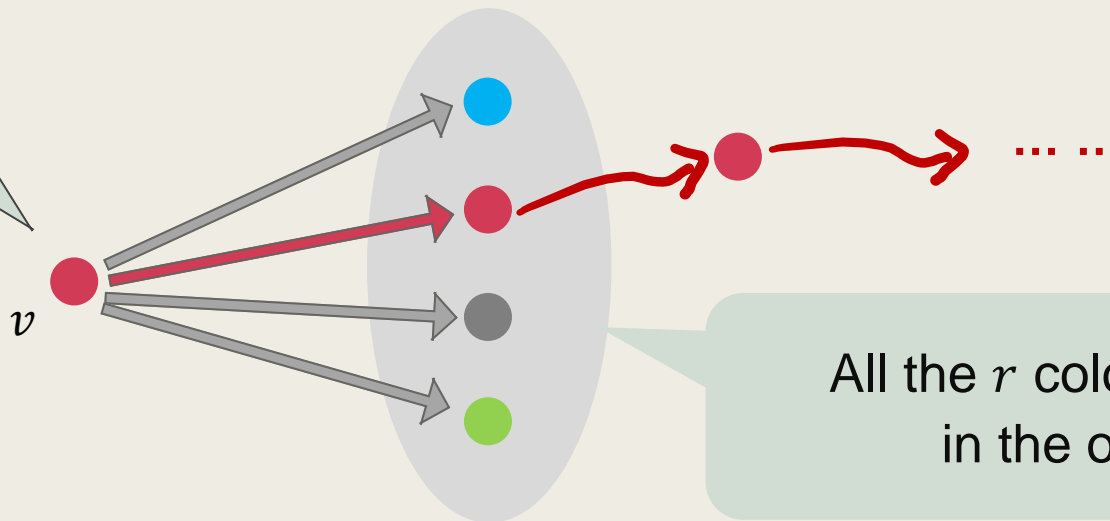
- Consider a random coloring of the vertices using $r := \lfloor k/(3 \ln k) \rfloor$ colors.

We will show that, there exists a coloring such that,

every vertex has all the r colors in its out-neighbors.

Continue to follow
the same color.

We will eventually
obtain a cycle.



All the r colors appear at least once
in the out-neighbor set of v

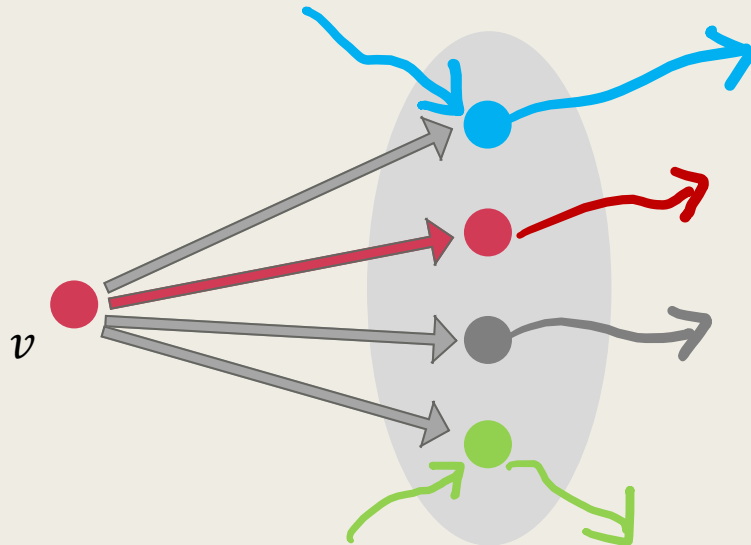
Theorem 19.4 (Erdős 1963a).

Every finite k -regular directed graph has a collection of $\lfloor k/(3 \ln k) \rfloor$ vertex disjoint cycles.

- Consider a random coloring of the vertices using $r := \lfloor k/(3 \ln k) \rfloor$ colors.

We will show that, there exists a coloring such that,

every vertex has all the r colors in its out-neighbors.



Apply the same argument for all colors.

This will imply the conclusion of the theorem.

- Consider a random coloring of the vertices using $r := \lfloor k/(3 \ln k) \rfloor$ colors.
- For any $v \in V$, let A_v denote the event that
not every color is used in the out-neighbors of v .
 - For any $1 \leq i \leq r$, let $A_{i,v}$ denote the event that
the i^{th} color is not used in the out-neighbors of v .
 - Then we obtain

$$\Pr[A_v] = \Pr\left[\bigcup_{1 \leq i \leq r} A_{i,v}\right] \leq r \cdot \left(1 - \frac{1}{r}\right)^k \leq r \cdot e^{-\frac{k}{r}} \leq \frac{1}{3k^2 \ln k}.$$

- Consider a random coloring of the vertices using $r := \lfloor k/(3 \ln k) \rfloor$ colors.

- For any $v \in V$, let A_v denote the event that
not every color is used in the out-neighbors of v .

- Then we obtain

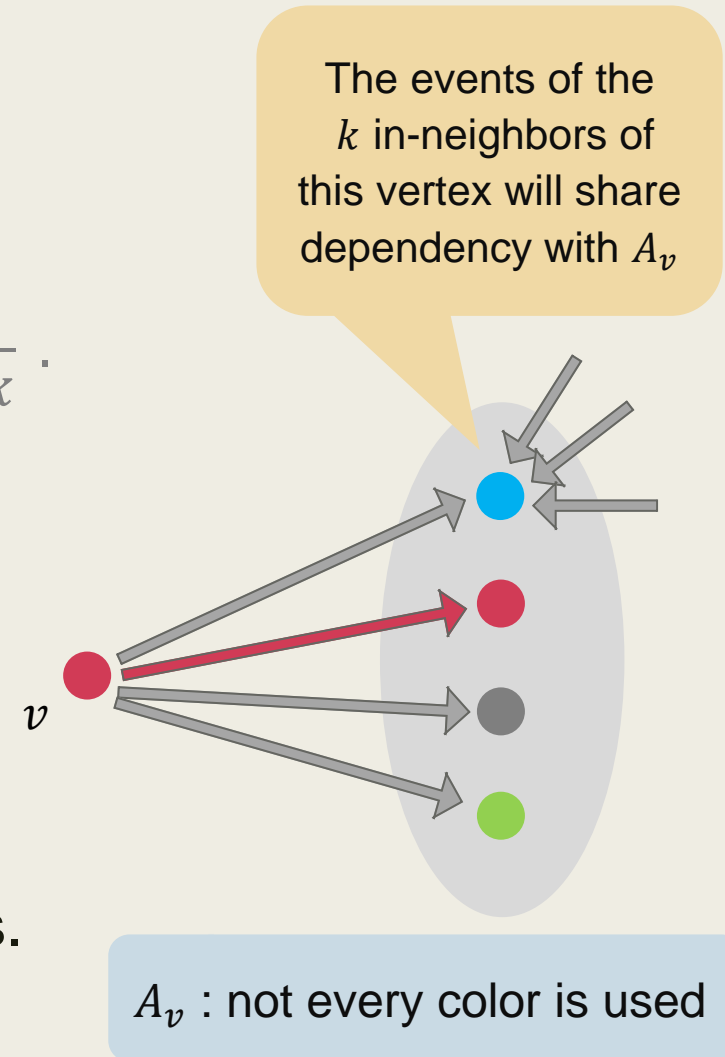
$$\Pr[A_v] \leq r \cdot \left(1 - \frac{1}{r}\right)^k \leq r \cdot e^{-\frac{k}{r}} \leq \frac{1}{3k^2 \ln k}.$$

- Let $N(v)$ denote the out-neighbors of v .

Then, A_v is mutually independent to

$$\{A_u : N(u) \cap N(v) = \emptyset\}.$$

So, A_v shares dependency with at most k^2 events.



- Consider a random coloring of the vertices using $r := \lfloor k/(3 \ln k) \rfloor$ colors.
- For any $v \in V$, let A_v denote the event that not every color is used in the out-neighbors of v .
 - Then $\Pr[A_v] \leq r \cdot \left(1 - \frac{1}{r}\right)^k \leq r \cdot e^{-\frac{k}{r}} \leq \frac{1}{3k^2 \ln k}$.
 - The maximum degree of dependency $d \leq k^2$.
 - Since
$$e \cdot \frac{1}{3k^2 \ln k} \cdot (k^2 + 1) \leq 1, \quad \forall k \geq 3,$$
$$\Pr[\cap \overline{A_v}] > 0$$
 by the Lovász Local Lemma.
 - Hence, there must exist such a good coloring for $k \geq 3$.
 - For $k \leq 2$, $r = 0$ and the statement of the lemma holds trivially.

Some remarks

- If you have read the textbook,
the proof of the Theorem 19.4 in the textbook is incorrect.
 - The reason is that, it sets the event A_v as
“the color of v does not appear in the out-neighbors of v .”
 - As a result, it didn’t consider the event that not all the r colors are used.
 - When this happens, we don’t get r disjoint cycles, and
what it has truly proved is that,
“*there exists a cycle in the graph.*”

The Lovász Local Lemma

--- The Asymmetric Version

When the universal bounds of the events are not good enough.

Theorem 19.2.

Let $G = (V, E)$ be a dependency graph of events A_1, A_2, \dots, A_n .

Suppose that there exists real numbers x_1, x_2, \dots, x_n with $0 \leq x_i < 1$ such that, for all i ,

$$\Pr[A_i] \leq x_i \cdot \prod_{j:(i,j) \in E} (1 - x_j) .$$

Then

$$\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] \geq \prod_{1 \leq i \leq n} (1 - x_i) .$$

In particular, with positive probability, no A_i occurs.

Ex2.

2-Colorable Families

2-Colorable Families

- In Lecture 2, we use simple union bound to show that when the size of a k -uniform family is no more than 2^{k-1} , it is 2-colorable.
- We use the Lovász Local Lemma to prove a stronger statement, which takes the structure of the family into consideration.

It says that, when the dependency of the members is bounded within 2^{k-3} , the family is 2-colorable.

Theorem 19.5 (Erdős-Lovász 1975).

If every member of a k -uniform family intersects at most 2^{k-3} other members, then the family is 2-colorable.

Theorem 19.5 (Erdős-Lovász 1975).

If every member of a k -uniform family $F = \{S_1, S_2, \dots, S_m\}$ intersects at most 2^{k-3} other members, then the family is 2-colorable.

Proof.

- Let X be the ground set. Consider a random 2-coloring on X and let A_i be the event that S_i is monochromatic.
 - We have $\Pr[A_i] = 2 \cdot 2^{-|S_i|} = 2^{1-k}$.
 - Any A_i is mutually independent to $\{A_j : S_i \cap S_j = \emptyset\}$.
So the maximum degree of dependency $d \leq 2^{k-3}$.
- Since $4pd = 2^0 \leq 1$, the LLL guarantees that $\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] > 0$.

- For non-uniform families, we have the following theorem.

Theorem 19.6 (Beck 1980).

Let $F = \{S_1, S_2, \dots, S_m\}$ be a family of sets, each of which has at least $k \geq 2$ elements. Suppose that for each element v in the ground set,

$$\sum_{S \in F: v \in S} \left(1 - \frac{1}{k}\right)^{-|S|} \cdot 2^{-|S|+1} \leq \frac{1}{k}.$$

Then F is 2-colorable.

- Consider a random coloring, and let A_i be the event that S_i is monochromatic. Include an edge for A_i, A_j in the dependency graph if and only if $S_i \cap S_j \neq \emptyset$, and define $x_i := \left(1 - \frac{1}{k}\right)^{-|S_i|} \cdot 2^{-|S_i|+1}$ for all i .

Theorem 19.6 (Beck 1980).

Let $F = \{S_1, S_2, \dots, S_m\}$ be a family of sets, each of which has at least $k \geq 2$ elements.

Suppose that for each element v , $\sum_{S \in F: v \in S} \left(1 - \frac{1}{k}\right)^{-|S|} \cdot 2^{-|S|+1} \leq \frac{1}{k}$.

Then F is 2-colorable.

- Consider a random coloring, and let A_i be the event that S_i is monochromatic. Include an edge for A_i, A_j in the dependency graph if and only if $S_i \cap S_j \neq \emptyset$, and define $x_i := \left(1 - \frac{1}{k}\right)^{-|S_i|} \cdot 2^{-|S_i|+1}$ for all i .
- To apply the local lemma, we need to show that

$$x_i \cdot \prod_{j: (i,j) \in E} (1 - x_j) \geq \Pr[A_i], \quad \forall 1 \leq i \leq m.$$

Theorem 19.6 (Beck 1980).

Let $F = \{S_1, S_2, \dots, S_m\}$ be a family of sets, each of which has at least $k \geq 2$ points.

Suppose that for each point v , $\sum_{S \in F: v \in S} \left(1 - \frac{1}{k}\right)^{-|S|} \cdot 2^{-|S|+1} \leq \frac{1}{k}$.

Then F is 2-colorable.

Define $x_i := \left(1 - \frac{1}{k}\right)^{-|S_i|} \cdot 2^{-|S_i|+1}$ for all i .

■ We have

$$0 \leq x_i < 1$$

$$x_i \cdot \prod_{j:(i,j) \in E} (1 - x_j) \geq x_i \cdot \prod_{v \in S_i} \prod_{j:v \in S_j} (1 - x_j)$$

Interpret x_i as probabilities of some other independent events. Refer to jamboard for details.

$$\geq x_i \cdot \prod_{v \in S_i} \left(1 - \sum_{j:v \in S_j} x_j\right) \geq x_i \cdot \left(1 - \frac{1}{k}\right)^{|S_i|}$$

by the definition of x_i

$$= 2^{-|S_i|+1} = \Pr[A_i] .$$

by the assumption of the theorem

Can we actually construct the object ?

Some remark

- The Lovász Local Lemma, and the probabilistic method we introduced, aims to prove the existence of an object satisfying a set of constraints.
 - A natural question is that,
can we actually *compute* such an object *efficiently*?
 - When the conditions in the Lovász Local Lemma are met, the answer is **yes!**

Such an object can be constructed in expected $O\left(\sum_A \frac{x(A)}{1-x(A)}\right)$ time.

We will talk about this in lecture #15 (as supplementary content).

Proof of Theorem 19.1

(Symmetric LLL, weaker version)

Proof of the LLL (weaker version)

- We will prove the theorem under a slightly stronger condition, i.e., $4pd \leq 1$.

Theorem (Erdős-Lovász 1975).

Let A_1, A_2, \dots, A_n be events with $\Pr[A_i] \leq p$ for all i , and let d be the maximum degree of their dependence.

If $4pd \leq 1$, then $\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] > 0$.

- In HW3, you will use asymmetric LLL to prove the stronger version of symmetric LLL with $ep(d + 1) \leq 1$.

Tools to Use

- We will use the following two identities for conditional probability.

- $\Pr[A \mid BC] = \frac{\Pr[AB \mid C]}{\Pr[B \mid C]} .$

- $\Pr[ABC] = \Pr[A \mid BC] \cdot \Pr[B \mid C] \cdot \Pr[C] .$

They follow directly from the definition of conditional probability.

Tools to Use

- In general,

$$\Pr[A \mid B_1 B_2 \cdots B_m] = \frac{\Pr[AB_1 B_2 \cdots B_j \mid B_{j+1} B_{j+2} \cdots B_m]}{\Pr[B_1 B_2 \cdots B_j \mid B_{j+1} B_{j+2} \cdots B_m]} \quad \forall 1 \leq j \leq m,$$

(*)

and

$$\Pr[A_1 A_2 \cdots A_m] = \prod_{1 \leq j \leq m} \Pr[A_j \mid A_{j+1} A_{j+2} \cdots A_m]$$

(**)

Refer to the jamboard for the details.

Theorem (Erdős-Lovász 1975).

Let A_1, A_2, \dots, A_n be events with $\Pr[A_i] \leq p$ for all i , and let d be the maximum degree of their dependence. If $4pd \leq 1$, then $\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] > 0$.

Proof.

- Fix a dependency graph with maximum degree d .
- We will prove that, for **any subset of events** of A_1, A_2, \dots, A_n , denoted B_1, B_2, \dots, B_m for convenience,

we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq 2p.$$

- We will prove that, for **any subset of m events** of A_1, A_2, \dots, A_n , denoted B_1, B_2, \dots, B_m for convenience,

we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq 2p.$$

- If this holds, then by (**), we have

$$\begin{aligned} \Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] &= \prod_{1 \leq j \leq n} \Pr[\overline{A_j} \mid \overline{A_{j+1}} \overline{A_{j+2}} \cdots \overline{A_n}] \\ &= \prod_{1 \leq j \leq n} (1 - \Pr[A_j \mid \overline{A_{j+1}} \overline{A_{j+2}} \cdots \overline{A_n}]) \geq (1 - 2p)^n > 0. \end{aligned}$$

It suffices to show that, for any subset of m events of A_1, A_2, \dots, A_n , denoted B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq 2p.$$

- Prove by induction on m .
 - The base case $m = 1$ is trivial.
 - For $m \geq 2$, assume without loss of generality that, B_1 and B_{k+1}, \dots, B_m are mutually independent.
 - i.e., B_1 share dependency only with B_2, B_3, \dots, B_k .

Hence, $k - 1 \leq d$.

It suffices to show that, for any subset of m events of A_1, A_2, \dots, A_n , denoted B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq 2p.$$

- For $m \geq 2$, assume without loss of generality that, B_1 and B_{k+1}, \dots, B_m are mutually independent.

Hence, $k - 1 \leq d$.

■ i.e., B_1 share dependency only with B_2, B_3, \dots, B_k .

- By (*), we have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] = \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}$$

Consider the numerator and the denominator separately.

- Assume that B_1 is mutually independent to B_{k+1}, \dots, B_m .

By (*), we have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] = \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]} .$$

- For the numerator, we have

$$\begin{aligned} \Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}] &\leq \Pr[B_1 \mid \overline{B_{k+1}} \cdots \overline{B_m}] \\ &= \Pr[B_1] \leq p . \end{aligned}$$

Since B_1 is mutually independent of B_{k+1}, \dots, B_m

It suffices to show that, for any subset of m events of A_1, A_2, \dots, A_n , denoted B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq 2p.$$

- For the denominator,

$$\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}] = 1 - \Pr[B_2 \cup \cdots \cup B_k \mid \overline{B_{k+1}} \cdots \overline{B_m}]$$

Union bound

$$\geq 1 - \sum_{2 \leq i \leq k} \Pr[B_i \mid \overline{B_{k+1}} \cdots \overline{B_m}]$$

Induction hypothesis

$$\geq 1 - 2p(k-1) \geq \frac{1}{2},$$

since $2p(k-1) \leq 2pd \leq 1/2$.

Instead of applying union bound directly, this lemma applies when the events are properly conditioned.

It suffices to show that, for any subset of m events of A_1, A_2, \dots, A_n , denoted B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq 2p.$$

■ Then, we obtain

$$\begin{aligned} \Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] &= \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]} . \\ &\leq \frac{p}{1/2} = 2p . \end{aligned}$$

Proof of the Asymmetric LLL

Theorem 19.2.

Let $G = (V, E)$ be a dependency graph of events A_1, A_2, \dots, A_n .

Suppose that there exists real numbers x_1, x_2, \dots, x_n with $0 \leq x_i < 1$ such that, for all i ,

$$\Pr[A_i] \leq x_i \cdot \prod_{j:(i,j) \in E} (1 - x_j) .$$

Then

$$\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] \geq \prod_{1 \leq i \leq n} (1 - x_i) .$$

In particular, with positive probability, no event A_i holds.

- The proof is analogous to the symmetric version of the lemma.
- We will use induction to prove that,

for any subset of events of A_1, A_2, \dots, A_n , say, B_1, B_2, \dots, B_m , for convenience, we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq x_1 .$$

Then by (**) we have

$$\Pr[\overline{A_1} \overline{A_2} \cdots \overline{A_n}] = \prod_{1 \leq j \leq n} \left(1 - \Pr[A_j \mid \overline{A_{j+1}} \overline{A_{j+2}} \cdots \overline{A_n}] \right) \geq \prod_{1 \leq i \leq n} (1 - x_i) .$$

- The induction base $m = 1$ follows from the assumption of the lemma.
For $m \geq 2$, we consider an arbitrary combination of m events.

It suffices to show that,

for any subset of m events of A_1, A_2, \dots, A_n , say, B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq x_1.$$

- W.L.O.G., let B_2, B_3, \dots, B_k be events that share dependency with B_1 , while B_{k+1}, \dots, B_m are mutually independent to B_1 .

- By (*), we have $\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] = \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}.$

- For the numerator,

$$\begin{aligned} \Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}] &\leq \Pr[B_1 \mid \overline{B_{k+1}} \cdots \overline{B_m}] = \Pr[B_1] \\ &\leq x_1 \cdot \prod_{j:(i,j) \in E} (1 - x_j) \leq x_1 \cdot \prod_{2 \leq j \leq k} (1 - x_j) . \end{aligned}$$

It suffices to show that,

for any subset of m events of A_1, A_2, \dots, A_n , say, B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq x_1.$$

- W.L.O.G., let B_2, B_3, \dots, B_k be events that share dependency with B_1 , while B_{k+1}, \dots, B_m are mutually independent to B_1 .

- By (*), we have $\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] = \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}.$

- For the denominator, apply (**) and the induction hypothesis, we obtain

$$\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}] = \prod_{2 \leq j \leq k} \Pr[\overline{B_j} \mid \overline{B_{j+1}} \cdots \overline{B_m}] \geq \prod_{2 \leq j \leq k} (1 - x_j) .$$

It suffices to show that,

for any subset of m events of A_1, A_2, \dots, A_n , say, B_1, B_2, \dots, B_m , we always have

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] \leq x_1.$$

- W.L.O.G., let B_2, B_3, \dots, B_k be events that share dependency with B_1 , while B_{k+1}, \dots, B_m are mutually independent to B_1 .

- By (*), we have $\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] = \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}.$

- Combine the two inequalities. We obtain

$$\Pr[B_1 \mid \overline{B_2} \overline{B_3} \cdots \overline{B_m}] = \frac{\Pr[B_1 \overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}{\Pr[\overline{B_2} \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]} \leq \frac{x_1 \cdot \prod_{2 \leq j \leq k} (1 - x_j)}{\prod_{2 \leq j \leq k} (1 - x_j)} = x_1 .$$

Some remark

- In HW3, you will prove that Theorem 19.2 leads to Theorem 19.1.
 - This is done as follows.

Set $x_i = \frac{1}{d+1}$ for each event A_i , and apply the inequality that

$$\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d.$$

This can be obtained from the limit formula $e = \lim_{d \rightarrow \infty} \left(1 + \frac{1}{d}\right)^d$ and the fact that it converges from the above.