Lemma. if A is mutually independent of $B = \{B_1, B_2 \cdots B_n\}$.

then A is " " of any $C \subseteq B$.

Pf. Fix a subset $C \subseteq B$. Let $C' = B \setminus C$.

Let D be a Boolean combination of C.

Then $$\Pr[A \cap D] = \sum_{D' :\, D' \text{ is a Boolean Combination of } C'} \Pr[A \cap D \cap D']$$

Any $D \cap D'$ is a Boolean combination of B

$D' \cap D'' = \phi$. $\forall$ Boolean Combination D', D'' of C'.

and $\displaystyle\bigsqcup_{D' :\, \text{Boolean Combi of } C'} D'$ = The universe

$$= \sum_{D' \text{ Boolean combination of } C'} \Pr[A] \cdot \Pr[D \cap D']$$

$$= \Pr[A] \cdot \Pr[D]. \ \#$$

Let $A_1, A_2, \cdots, A_k$ be ~~events~~ $\boxed{\text{independent events}}$ with probabilities $x_1, x_2, \cdots, x_k$, $0 \le x_i \le 1$. $\forall\, 1 \le i \le k$.

Then.

$$\prod_{1\le i\le k}(1-x_i) \qquad \Pr[\bar{A}_1 \cdot \bar{A}_2 \cdots \bar{A}_k] = 1 - \Pr[A_1 \cup A_2 \cup \cdots \cup A_k]$$

$$\prod_{1\le i\le k}(1-x_i) = \prod_{1\le i\le k}\Pr[\bar{A}_i] \overset{\boxed{\text{independence}}}{=} \Pr[\bar{A}_1 \cdot \bar{A}_2 \cdots \bar{A}_k]$$

$$1 - \Pr[A_1 \cup A_2 \cup \cdots \cup A_k] \overset{\boxed{\text{Union bound}}}{\geqslant} 1 - \sum_{1\le i\le k} x_i \, .$$

Method 2. $\boxed{\text{by induction.}}$ $\qquad 0 \le x_i \le 1$. $\forall\, 1 \le i \le k$.

(i). $1 - x_1 \geqslant 1 - x_1$.

(ii)

$$\prod_{1\le i\le k}(1-x_i) = (1-x_1)\cdot\prod_{2\le i\le k}(1-x_i)$$

$$\overset{\boxed{\text{induction hypothesis}}}{\geqslant} (1-x_1)\Big(1-\sum_{2\le i\le k}x_i\Big)$$

$$= 1 - \sum_{1\le i\le k}x_i + x_1\cdot\sum_{2\le i\le k}x_i \geqslant 1 - \sum_{1\le i\le k}x_i \, .$$

$$\forall A, B. \quad \Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}.$$

$$\Rightarrow \Pr[A \cap B] = \Pr[A \mid B] \cdot \Pr[B].$$

The multiplication rule

For (**),

$$
\begin{aligned}
\Pr[A_1 A_2 \cdots A_m] &= \Pr[(A_1 A_2 \cdots A_{m-1}) \cdot A_m] \\
&= \Pr[A_1 A_2 \cdots A_{m-1} \mid A_m] \cdot \Pr[A_m] \\
&= \Pr[(A_1 A_2 \cdots A_{m-2}) \cdot A_{m-1} \mid A_m] \cdot \Pr[A_m] \\
&= \Pr[A_1 A_2 \cdots A_{m-2} \mid A_{m-1} A_m] \cdot \Pr[A_{m-1} \mid A_m] \cdot \Pr[A_m] \\
&\ \ \vdots \\
&= \prod_{1 \le j \le m} \Pr[A_j \mid A_{j+1} A_{j+2} \cdots A_m].
\end{aligned}
$$

$$\forall A, B. \quad Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}.$$

$$\Rightarrow Pr[A \cap B] = Pr[A \mid B] \cdot Pr[B].$$

The multiplication rule

Similarly, for (*).

$$Pr[A \mid B_1 B_2 \cdots B_m] = \frac{Pr[A B_1 \mid B_2 \cdots B_m]}{Pr[B_1 \mid B_2 \cdots B_m]}$$

$$= \frac{Pr[A \cdot B_1 \cdot B_2 \mid B_3 \cdots B_m]}{Pr[B_1 B_2 \mid B_3 \cdots B_m]}$$

$$\vdots$$

$$= \frac{Pr[A \cdot B_1 B_2 \cdots B_m]}{Pr[B_1 B_2 \cdots B_m]}.$$