Union Bound

Let $A_1, A_2, \cdots, A_n$ be events.

Then $\Pr[\bigcup_{1\le i\le n} A_i] \le \sum_{1\le i\le n} \Pr[A_i]$.

Two useful inequalities

* $\forall t \neq 0, \quad 1+t < e^t$.

  Proved by Taylor's expansion on e^t.

* $\boxed{1-t > e^{-t-t^2} \quad \forall\, 0<t<0.6838\ldots}$

  by Taylor's expansion on $\ln(1-t)$.

  See P.4 for more details.

  Note that Inequality (1.5) in the textbook is incorrect and should be updated.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \le \frac{n^k}{k!}$$

$$(1-2^{-k})^{n-k} < \left(e^{-2^{-k}}\right)^{n-k} = e^{-\frac{n-k}{2^k}}$$

$$\Rightarrow \Pr[\bigcup A_s] < \frac{n^k}{k!} \cdot e^{-\frac{n-k}{2^k}} \le n^k \cdot e^{-\frac{n}{2^k}}$$

Since $\frac{1}{k!} \cdot e^{\frac{k}{2^k}} \le 1 \quad \forall\, k \ge 2$.

To require $n^k \cdot e^{-\frac{n}{2^k}} < 1$.

we need $k \cdot \log n - \frac{n}{2^k} < 0$.

$\Rightarrow n > k \cdot 2^k \cdot \log n$.

When $n \ge k^2 \cdot 2^{k+1}$, the recursion is satisfied.

$\boxed{\text{See P.4 for more details}}$

Stirling formula for the factorial

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot e^{\alpha_n}.$$

where $\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$.

This is a very tight approximation for $n!$

Jensen's Inequality

If $\lambda_i \geq 0$, $\sum_{1\le i\le n} \lambda_i = 1$, and f is convex, then

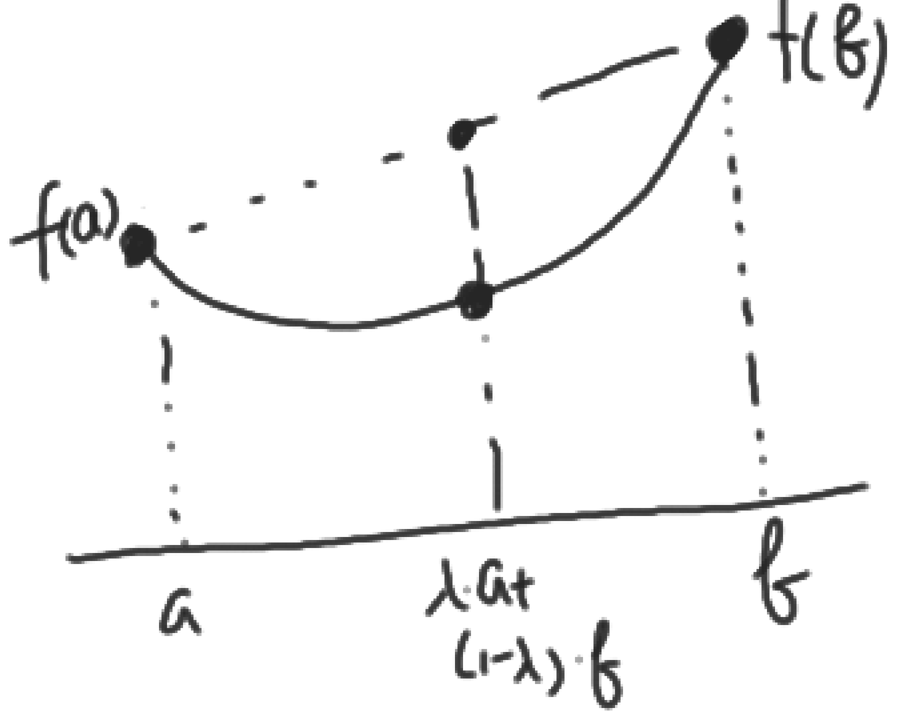
$$f\left(\sum_{1\le i\le n} \lambda_i x_i\right) \le \sum_{1\le i\le n} \lambda_i f(x_i).$$

Convex function

A real-valued function $f(x)$ is convex if

$$f(\lambda a + (1-\lambda)\cdot b) \le \lambda \cdot f(a) + (1-\lambda)\cdot f(b).$$

$\forall\, 0 \le \lambda \le 1.$



Proof. for $n=2$. holds by def.

for $n \geq 3$.

write $\sum_{1\le i\le n} \lambda_i \cdot x_i$ as

$$(\lambda_1+\lambda_2)\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2} x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} x_2\right) + \sum_{3\le i\le n} \lambda_i x_i.$$

Apply the induction hypothesis.

The Stirling formula leads to the following useful aymptotic formula for k-th factorial

$$(n)_k = n\cdot(n-1)\cdots(n-k+1)$$
$$= n^k\cdot e^{-\frac{k^2}{2n}-\frac{k^3}{6n^2}+o(1)} \quad \forall k = o(n^{\frac{3}{4}}).$$

<u>Fact</u>. $\binom{r}{k}$ is convex $\forall\, r \geq k$.

$$\Rightarrow \frac{1}{2}\binom{a}{k} + \frac{1}{2}\binom{r-a}{k} \geq \binom{\frac{1}{2}r}{k}$$ by Jensen's Inequality

$$\Rightarrow \binom{a}{k} + \binom{r-a}{k} \geq 2\cdot\binom{\frac{1}{2}r}{k}$$

By the formula in the left.

$$2\cdot\frac{\binom{\frac{r}{2}}{k}}{\binom{r}{k}} = 2\cdot\frac{\left(\frac{r}{2}\right)_k}{(r)_k}$$
$$\approx 2\cdot\left(\frac{1}{2}\right)^k\cdot\frac{e^{-\frac{k^2}{r}+o(1)}}{e^{-\frac{k^2}{2r}+o(1)}}$$

$$\boxed{r = \frac{1}{2}k^2}$$

$$\approx 2^{1-k}\cdot\frac{e^{-2}}{e^{-1}} = 2^{1-k}\cdot e^{-1}$$

$$\boxed{n > k\cdot 2^k \log n \text{ when } n \geq k^2\cdot 2^{k+1}}$$

when $n = k^2\cdot 2^{k+1}$.

we have $k\cdot 2^k \log n = k\cdot 2^k\cdot(\underbrace{2\log k + (k+1)\log 2}_{<2k}) < k^2\cdot 2^{k+1}$.

when the value of n grows.

the R.H.S ($k\cdot 2^k \log n$) grows slower than the L.H.S (n).

$$\ln(1-x) = -\sum_{n\geq 1}\frac{x^n}{n} = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$$

Converges when $|x| < 1$.

$$\Rightarrow 1-x = e^{-x-x^2+\left(\frac{1}{2}x^2 - \sum_{n\geq 3}\frac{1}{n}x^n\right)} > e^{-x-x^2}.$$

$\forall\, 0 < x < 0.6$. since $\boxed{\frac{1}{2}x^2 - \sum_{n\geq 3}\frac{1}{n}x^n > 0. \text{ for } 0<t<0.6838\ldots}$

MATLAB says it holds. I do not know why.
(Tell me if you do.)