Combinatorial Mathematics

Mong-Jen Kao (高孟駿)

Monday 18:30 – 21:20

Outline

- Adjacency Matrix & Random Walks in Graphs
- Eigenvalue & Spectral Gap
- Expander Graph
  - Algebraic Expansion v.s. Edge Expansion
  - Expander & Pseudo-randomness
  - Explicit Constructions

Random Walks in Graphs

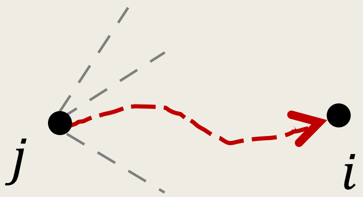
Let’s take a random stroll in the graph.

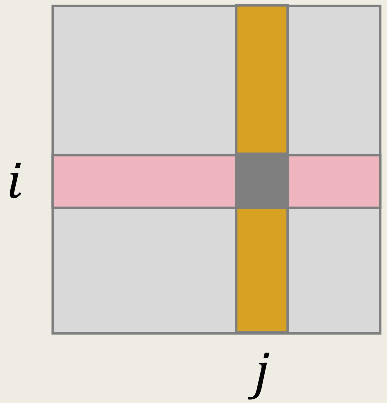
Where will we be after a number of steps?

The Normalized Adjacency Matrix

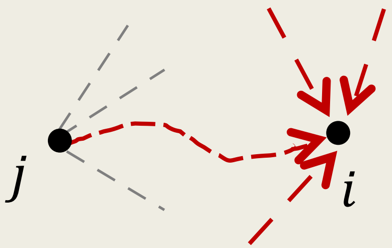
- Let $G = (V, E)$ be an n-vertex d-regular graph.
- Let A^* be the adjacency matrix of G and define $A \coloneqq A^*/d$.
  - The sum of *each row* in A is 1.
  - Think $\boldsymbol{a_{i,j}}$ as the ***probability** that*

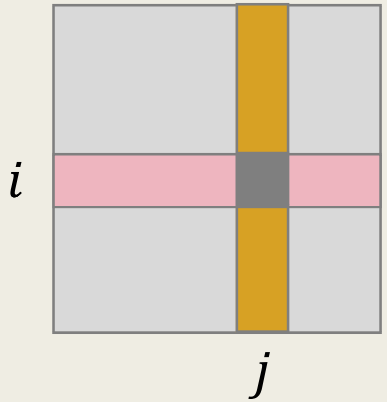
    we move to vertex i when we are at vertex j.



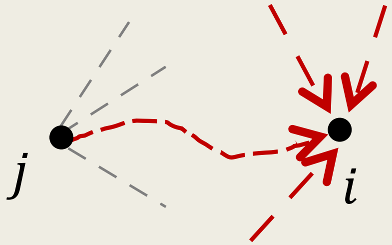


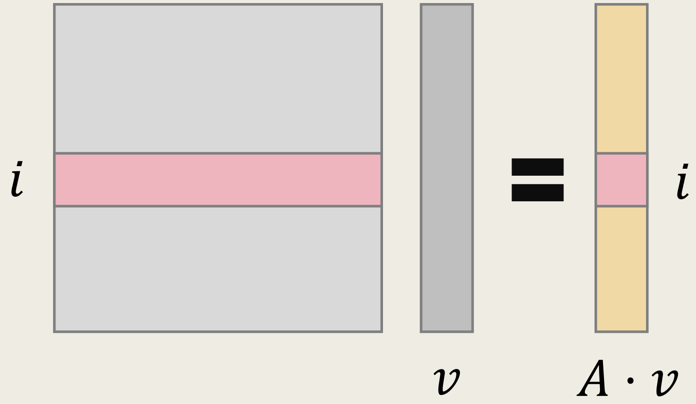
- Let A^* be the adjacency matrix of G and define $A \coloneqq A^*/d$.
  - Think $\boldsymbol{a_{i,j}}$ as the ***probability** that* we move to vertex i when we are at vertex j.
  - Then, the i^{th}-row of A describes the probability that we reach vertex i from each vertex *in* V.





- Let A^* be the adjacency matrix of G and define $A := A^*/d$.
  - Think $\boldsymbol{a_{i,j}}$ as the ***probability** that* we move to vertex i when we are at vertex j.
  - Let $v = (p_1, p_2, \dots, p_n)^T$ be a probability distribution over V that denotes our starting point.
  - Then, $\boldsymbol{Av}$ gives the probability distribution of the location we will be in **1-*step of random walk***.





- Let A^* be the adjacency matrix of G and define $A \coloneqq A^*/d$.
  - Let $v = (p_1, p_2, \dots, p_n)^T$ be a probability distribution over V that denotes our starting point.
  - Then, Av gives the probability distribution of the location we will be in 1-*step of random walk*.
  - Similarly, $A^t v = A^{t-1}(Av)$ gives the probability distribution after t steps.
    - Question: Where will we be?
    - Intuitively, when $t \approx \infty$,
      $A^t v$ should be close to uniform when G is connected.

How fast does it converge?

Eigenvalue & Spectral Gap

It turns out that,
eigenvalue plays an essential role in many important concepts.

The Eigenvalues of the Matrix A

- Let $G = (V, E)$ be an n-vertex d-regular graph and A be the normalized adjacency matrix of G.

Uniform distribution.

  – Clearly,

    1 is an eigenvalue of A with eigenvector $\mathbf{1} = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right) \in \mathbb{R}^n$,

    i.e.,

$$A\vec{\mathbf{1}} = \vec{\mathbf{1}}.$$

  – Furthermore,

    it can be shown that $\lambda \le 1$ for any eigenvalue λ of A.

In fact, $\lambda \le \max_{i} \sum_j |A_{i,j}| \le 1$ for any eigenvalue λ of A.

A is *real symmetric*.
Hence, all the eigenvalues of A are *real* numbers.

Eigenvalues & Spectral Gap

- Let $G = (V, E)$ be an n-vertex d-regular graph and A be the normalized adjacency matrix of G.
  - Clearly, 1 is an eigenvalue of A with eigenvector $\mathbf{1} = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$, i.e., $A\vec{\mathbf{1}} = \vec{\mathbf{1}}$.
  - Furthermore, $\lambda \leq 1$ for any eigenvalue λ of A.
  - Let $\boldsymbol{\lambda_2}$ be the $\mathbf{2^{nd}}$***-largest eigenvalue*** of A.
    - The quantity $(1 - \lambda_2)$ is called the *spectral gap* of A.

Spectral gap provides a lot of information on the *connectivity* of the graph.

Eigenvalues & Spectral Gap

- We have the following lemma.

Lemma 1.

Let $G = (V, E)$ be a regular graph with 2^{nd}-largest eigenvalue λ_2 and $\boldsymbol{p}$ be a probability distribution over V.

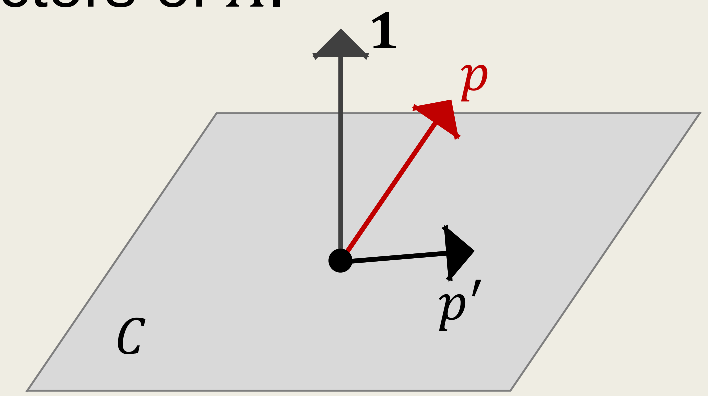
Then for any $\ell \in \mathbb{N}$,

$$\left\|A^{\ell}\boldsymbol{p} - \mathbf{1}\right\|_2 \leq (\lambda_2)^{\ell}.$$

L_2-norm

Proof of Lemma 1

- Recall that, $\mathbf{1} = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$ is an eigenvector of A with eigenvalue 1.
- Furthermore, we can obtain a set of orthonormal eigenvectors of A, including $\mathbf{1}$, that forms a basis of $\mathbb{R}^n$.
- Consider the subspace $C \subset \mathbb{R}^n$ that is orthogonal to $\mathbf{1}$.
  - C is spanned by the remaining eigenvectors of A.
- Rewrite the vector p as $p = p' + \alpha\mathbf{1}$, where $p' \in C$ and $\alpha \in \mathbb{R}$.



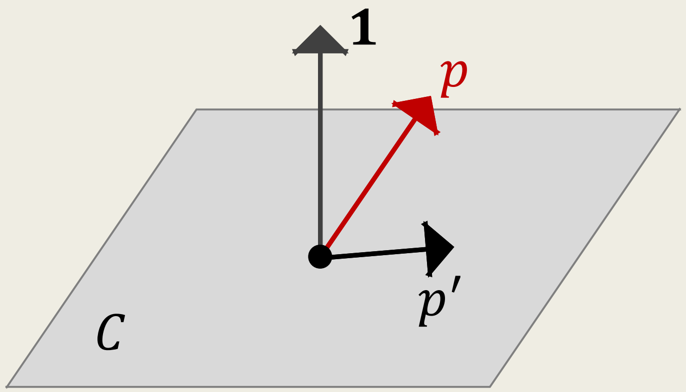
Proof of Lemma 1

- Consider the subspace $\mathcal{C} \subset \mathbb{R}^n$ that is orthogonal to $\mathbf{1}$.
  - C is spanned by the remaining eigenvectors of A.
- Write $p = p' + \alpha\mathbf{1}$, where $p' \in C$ and $\alpha \in \mathbb{R}$.
  - It follows that

$p' \in C$, hence $p' \perp \mathbf{1}$.

$$\frac{1}{n} \cdot \sum_i p_i = p \cdot \mathbf{1} = (p' + \alpha\mathbf{1}) \cdot \mathbf{1} = \frac{1}{n} \cdot \alpha\,.$$

  - Since p is a probability distribution, $\sum_i p_i = 1$ and hence $\alpha = 1$.



Proof of Lemma 1

- Write $p = p' + \alpha\mathbf{1}$, where $p' \in C$ and $\alpha \in \mathbb{R}$.
  - It follows that $\alpha = 1$.
- Hence,

$$\| A^{\ell} p - \mathbf{1} \|_2 = \| A^{\ell}(p' + \mathbf{1}) - \mathbf{1} \|_2 = \| A^{\ell} p' \|_2 .$$

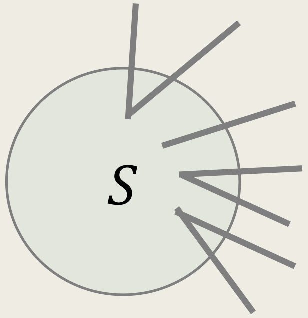
- Since λ_2 is the largest eigenvalue other than 1, we obtain

$$\left\| A^{\ell} p' \right\|_2 \leq \lambda_2^{\ell} \|p'\|_2 \leq \lambda_2^{\ell} \|p\|_2 \leq \lambda_2^{\ell} |p|_1 = \lambda_2^{\ell}.$$

$p \cdot p = p' \cdot p' + \mathbf{1} \cdot \mathbf{1}.$

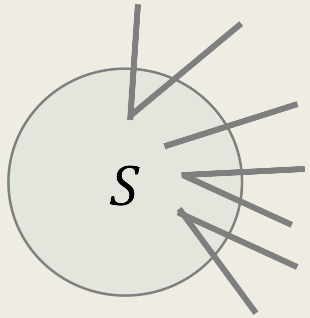
$\|p\|_2 \leq |p|$ for any vector p.

Expander Graph



For any subset of vertices with size at most $n/2$,
there are always *a lot of edges* "going out" from the subset.

Expander Graph



- Let $G = (V, E)$ be an n-vertex d-regular graph with 2^{nd}-largest eigenvalue λ_2.
  - Then,
    G is called an (n, d, λ)-expander graph for any $\lambda_2 \leq \lambda$.
  - We will show that,

    if G is an expander graph, then for any $S \subseteq V$ with $|S| \leq n/2$, there will be ***a lot of edges*** connecting S and $\bar{S}$.

Lemma 2. (Expander Crossing Lemma)

Let $G = (V, E)$ be an (n, d, λ)-expander and $S \subseteq V$, $T = V \setminus S$.

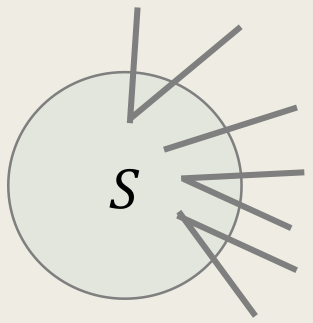
Then

$$|E(S,T)| \geq (1-\lambda) \cdot \frac{d|S||T|}{n},$$

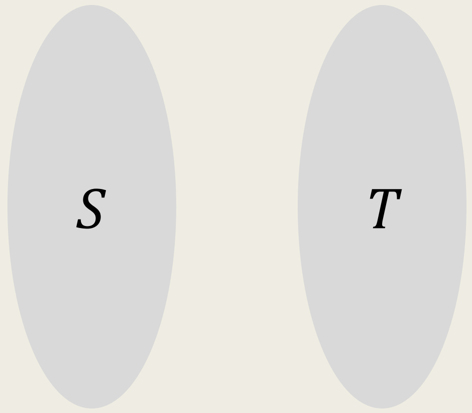
where $E(S,T)$ is the set of edges between S and T.

- In particular, when $|S| \leq n/2$,
  we have $|T| \geq n/2$ and

$$|E(S,T)| \geq \frac{d}{2}(1-\lambda)|S|.$$



Proof of Lemma 2



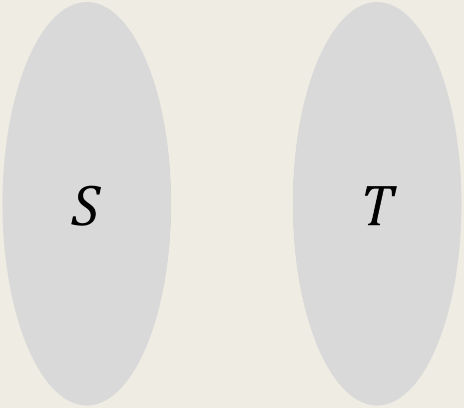
- Define the vector $x \in \mathbb{R}^{\mathrm{n}}$ as

$$x_i := \begin{cases} |T|, & \text{if } i \in S, \\ -|S|, & \text{if } i \in T. \end{cases}$$

Then, it follows that $x \perp \mathbf{1}$, and

$$\|x\|_2^2 \;=\; |S||T|^2 + |T||S|^2 \;=\; n \cdot |S||T| \;.$$

- Define the vector $x \in \mathbb{R}^\mathrm{n}$ as $x_i := \begin{cases} |T|, & \text{if } i \in S, \\ -|S|, & \text{if } i \in T. \end{cases}$



- On the other hand, define

$$Z := \sum_{i,j} A_{i,j} \left(x_i - x_j\right)^2 .$$

Then

  - Any $(i, j) \in E$ with $i \in S, j \in T$ appears twice in the summation, each contributing

$$\frac{1}{d}(|S| + |T|)^2 = \frac{1}{d} n^2 .$$

  - For the remaining cases, (i, j) contributes zero.

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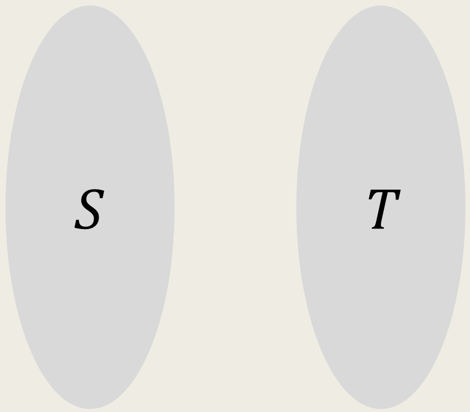

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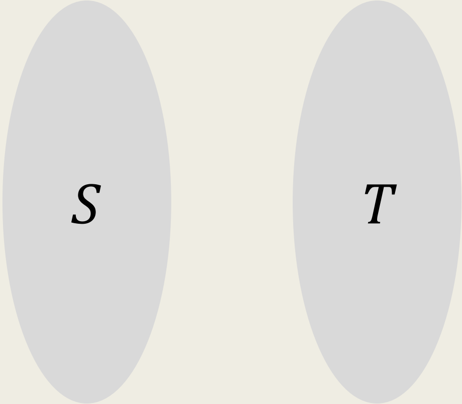
$$\frac{1}{d}(|S| + |T|)^2 = \frac{1}{d} n^2 .$$

  - For the remaining cases, (i,j) contributes zero.

- Hence,

$$Z = \frac{2}{d} \cdot |E(S,T)| \cdot n^2 .$$

- On the other hand,
  define $$Z := \sum_{i,j} A_{i,j}\left(x_i - x_j\right)^2 .$$



- On the other hand,
  expanding the summation in the above definition, we have

$$Z = \sum_{i,j} A_{i,j} x_i^2 - 2\sum_{i,j} A_{i,j} x_i x_j + \sum_{i,j} A_{i,j} x_j^2$$

$$= 2\|x\|_2^2 - 2 \cdot x \cdot Ax .$$

The rows and columns of A sum up to 1.

- Since $x \perp \mathbf{1}$,
  we obtain that $x \cdot Ax \leq \lambda \cdot \|x\|_2^2$.

- Hence,

$$Z = \frac{2}{d} \cdot |E(S,T)| \cdot n^2 .$$

- On the other hand, we have

$$Z = 2\|x\|_2^2 - 2 \cdot x \cdot Ax .$$

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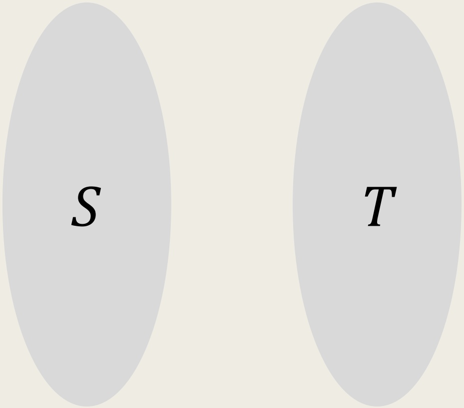


- Hence,

$$\frac{1}{d} \cdot |E(S,T)| \cdot n^2 \geq (1-\lambda) \cdot \|x\|_2^2 ,$$

and

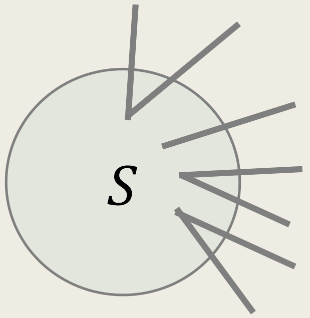
$$|E(S,T)| \geq (1-\lambda) \cdot \frac{d|S||T|}{n} .$$

$\|x\|_2^2 = n \cdot |S||T|.$

Connectivity of the Graph

- The expander crossing lemma implies that $G = (V, E)$ is connected if $\lambda_2 < 1$.
  - Indeed, for any $S \subset V$ and $T \coloneqq V \setminus S$,

$$|E(S,T)| \;\geq\; (1-\lambda) \cdot \frac{d|S||T|}{n} \;>\; 0\,.$$



  - The converse is also true, i.e., $\lambda_2 < 1$ if the G is connected.

Lemma 3.

Let $G = (V, E)$ be a d-regular graph with 2^{nd}-largest eigenvalue λ_2.
If G is connected, then $\lambda_2 < 1$.

- Suppose on the contrary that G is connected but $\lambda_2 = 1$.
  - Then, there exists $x \in \mathbb{R}^n$ such that

$$x \neq \mathbf{0}, \qquad x \cdot \mathbf{1} = 0, \qquad \text{and} \quad A \cdot x = x.$$

  - Pick i and j such that

$$x_i = \min_{1 \leq k \leq n} x_k \quad \text{and} \quad x_j = \max_{1 \leq k \leq n} x_k \quad .$$

Then,

$x_i < 0$ and $x_j > 0$.

- Suppose on the contrary that G is connected but $\lambda_2 = 1$.
  - Then, there exists $x \in \mathbb{R}^n$ such that

$$x \neq \mathbf{0}, \qquad x \cdot \mathbf{1} = 0, \qquad \text{and} \quad A \cdot x = x.$$

  - Pick i and j such that

$$x_i = \min_{1 \le k \le n} x_k \quad \text{and} \quad x_j = \max_{1 \le k \le n} x_k \quad .$$

Then,
$x_i < 0$ and $x_j > 0$.

  - Let $c := -1/(n \cdot x_i)$ and consider the vector $y := \mathbf{1} + cx$.

    Then

$$\boldsymbol{y} \geq 0, \qquad y_i = 0, \qquad \text{and} \quad y_j > 0.$$

Note that $c > 0$ by definition.

  - Furthermore,

$$A \cdot y \;=\; A \cdot \mathbf{1} + cA \cdot x \;=\; \mathbf{1} + cx \;=\; y.$$

- Suppose on the contrary that G is connected but $\lambda_2 = 1$.
  - Furthermore,

$$A \cdot y \;=\; A \cdot \mathbf{1} + cA \cdot x \;=\; \mathbf{1} + cx \;=\; y.$$

  - Then, $A^t \cdot y = y$.
  - Hence,

$$A_{i,j}^t \cdot y_j \;\le\; \sum_k A_{i,k}^t \cdot y_k \;=\; y_i \;=\; 0$$

which implies that $A_{i,j}^t = 0$ for all $t \in \mathbb{N}$.

- The following lemma says that, for arbitrarily $S, T \subseteq V$ that are <u>*sufficiently large*</u>, we have

$$|E(S,T)| \approx \frac{d}{n}|S||T| .$$

<u>Lemma 4. (Expander Mixing Lemma)</u>

Let $G = (V, E)$ be an (n, d, λ)-expander and $S, T \subseteq V$.

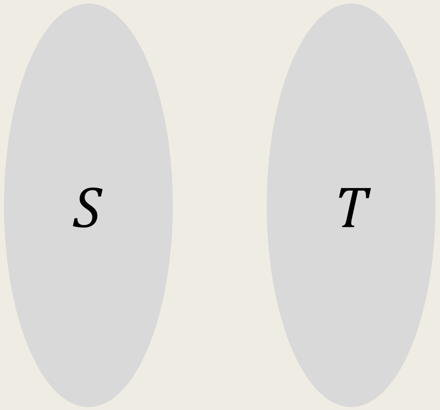
Then

$$\left| \, |E(S,T)| - \frac{d}{n}|S||T| \, \right| \le \lambda d\sqrt{|S||T|} ,$$

where $E(S,T)$ is the set of edges between S and T.

- Another interpretation of the expander mixing lemma is that,
  - λ measures **how close G behaves like a *random graph***.
  - To see this, observe that,
    - $|E(S,T)|$ is the number of edges between S and T.
    - $\frac{d}{n}|S||T|$ is the *expected number* of edges between S and T in a random graph, when the edge density is d/n.
  - Hence, when λ is small, the connectivity of G behaves like a random graph.

Connect each pair with probability $\frac{d}{n}$.



Proof of Lemma 4

- Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the normalized matrix A and $x_1 = \sqrt{n}\mathbf{1}, x_2, \ldots, x_n$ be the corresponding *orthonormal* eigenvectors.

- Let v_S and v_T be the characteristic vectors of S and T, i.e.,
  - The i^{th}-coordinate of v_S is 1 if and only if $i \in S$.
  - Express v_S and v_T as

$$v_S = \sum_i a_i x_i \quad \text{and} \quad v_T = \sum_i b_i x_i \,.$$

Since $\{x_i\}_{1 \leq i \leq n}$ forms a basis of $\mathbb{R}^n$.

- Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the normalized matrix A and $x_1 = \sqrt{n}\mathbf{1}, x_2, \dots, x_n$ the corresponding *orthonormal* eigenvectors.

- Let v_S and v_T be the characteristic vectors of S and T with

$$v_S = \sum_i a_i x_i \quad \text{and} \quad v_T = \sum_i b_i x_i \, .$$

- It follows that

$$\frac{|E(S,T)|}{d} = v_S^{\mathrm{T}} A v_T = \left(\sum_i a_i x_i\right)^{\mathrm{T}} A \left(\sum_i b_i x_i\right) = \sum_i \lambda_i a_i b_i \, .$$

$\{x_i\}_{1 \leq i \leq n}$ is an orthonormal basis.

- Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the normalized matrix A and $x_1 = \sqrt{n}\mathbf{1}, x_2, \ldots, x_n$ the corresponding *orthonormal* eigenvectors.
- Let v_S and v_T be the characteristic vectors of S and T with $v_S = \sum_i a_i x_i$ and $v_T = \sum_i b_i x_i$.
- It follows that $|E(S,T)| = d \cdot \sum_i \lambda_i a_i b_i$.
  - Furthermore, $a_1 = v_S \cdot x_1 = |S|/\sqrt{n}$ and $b_1 = |T|/\sqrt{n}$.
  - Hence, $\lambda_1 a_1 b_1 = |S||T|/n$.
  - $\lambda_i \leq \lambda$ for all $i \geq 2$.

  Hence $$\left|\sum_{i\geq 2} \lambda_i a_i b_i\right| \leq \lambda \cdot \left|\sum_{i\geq 2} a_i b_i\right| \leq \lambda \cdot \|a\|_2 \cdot \|b\|_2 .$$

By the Cauchy-Schwarz inequality.

- Let $x_1 = \sqrt{n}\mathbf{1}, x_2, \ldots, x_n$ be the *orthonormal* eigenvectors of A.
- Let v_S and v_T be the characteristic vectors of S and T with $v_S = \sum_i a_i x_i$ and $v_T = \sum_i b_i x_i$.
- It follows that

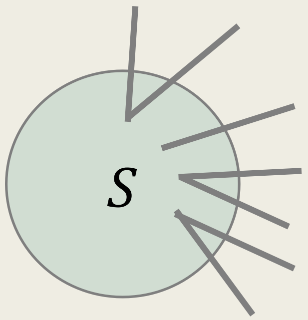
$$\left| |e(S,T)| - \frac{d|S||T|}{n} \right| = \left| \sum_{i \geq 2} \lambda_i a_i b_i \right| \leq \lambda d \cdot \|a\|_2 \cdot \|b\|_2.$$

- Since $\{x_i\}_{1 \leq i \leq n}$ is orthonormal,

$\|a\|_2 = \|v_S\|_2 = \sqrt{|S|}$ and $\|b\|_2 = \|v_T\|_2 = \sqrt{|T|}$, and

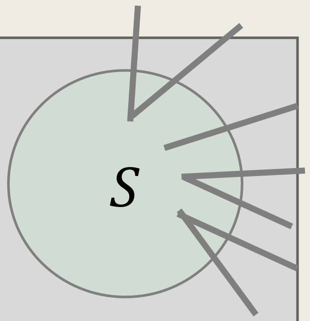
$$\left| |e(S,T)| - \frac{d|S||T|}{n} \right| \leq \lambda d \sqrt{|S||T|} \ .$$

Equivalent Notions



Edge expansion (Combinatorial expansion) is roughly equivalent to Algebraic expansion.

Definition. (Edge Expander)



Let $G = (V, E)$ be an n-vertex d-regular graph.

G is called an (n, d, ρ)-edge expander graph,

if for any vertex subset $S \subseteq V$ with $|S| \leq n/2$,

we always have

$$\left|E\left(S, \bar{S}\right)\right| \geq \rho d|S| .$$

- The expander crossing lemma says that, an (n, d, λ)-expander is also an edge expander with $\rho = (1 - \lambda)/2$.
  - The converse is roughly true as well.

Lemma 5. (Edge Expansion implies Algebraic Expansion)

Let $G = (V, E)$ be an (n, d, ρ)-edge expander.

Then, the 2^{nd}-largest eigenvalue of G is at most

$$1 - \rho^2/2 \,,$$

i.e., G is an (n, d, λ)-expander with $\lambda = 1 - \rho^2/2$.

- The proof, however, is beyond the scope of this course and is omitted here.

Expander Graph &

Pseudo-Randomness