Combinatorial Mathematics

Mong-Jen Kao (高孟駿)

Monday 18:30 – 21:20

Outline

- The Double-Counting Principle
  - The Handshaking Lemma
  - Average Number of Divisors
  - Turán Number
  - Pascal Triangle & Binomial Identities
  - Catalan Numbers
- The Density of 0-1 Matrices
- Principle of Inclusion and Exclusion

The Double Counting Principle

If the elements of a set are *counted in two different ways*, the answers are the same.

<u>Lemma 1.</u>

In any graph $G = (V, E)$,
the number of vertices with odd degrees is even.

- For each $v \in V$, let $d(v)$ denote the degree of v.

  Then, we have

$$\sum_{v \in V} d(v) = 2 \cdot |E|.$$

$2 \cdot |E|$ is an even number.

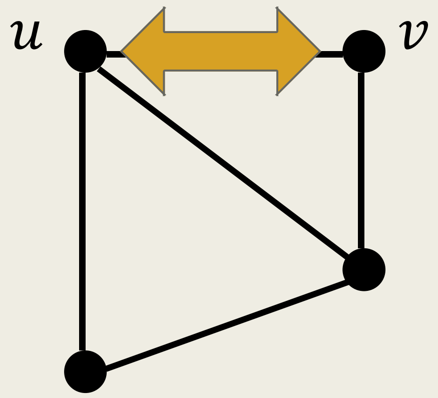
Each edge is counted exactly twice.

Hence, the number of vertices with odd degree must be even.

Handshaking Lemma.

At a party, the number of guests who shake hands an odd number of times is even.

- Consider the graph $G = (V, E)$ defined on the guests, where $(u, v) \in E$ if and only if guest u and v have shaken hands.



- Let F be a ***set family*** on a ground set X, i.e., F is a collection of subsets of X.
  - For any $x \in X$, let $d(x)$ be the number of sets in F that contain the element x, i.e., the degree of x in F.

The sets in F

$A_1 = \{e_2, e_4\}$

$A_3 = \{e_3, e_2, e_6\}$

$A_2 = \{e_1, e_5, e_7\}$

$A_4 = \{e_1, e_2, e_6, e_7\}$

The ground set X of elements

- e_1
- e_2
- $\vdots$
- e_n

- The previous identity, $\sum_v \deg(v) = 2|E|$, in Lemma 1 is a special case of the following general identity.

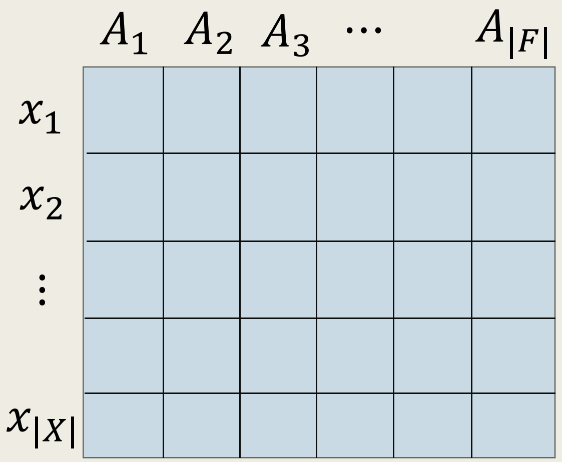
Proposition 2.

Let F be a family of subsets of some ground set X. Then

$$\sum_{x \in X} d(x) = \sum_{A \in F} |A| \ .$$

- Consider the $|X| \times |F|$ incidence matrix $M = (m_{x,A})$, where

$$m_{x,A} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$



The matrix M

<u>Proposition 2.</u>

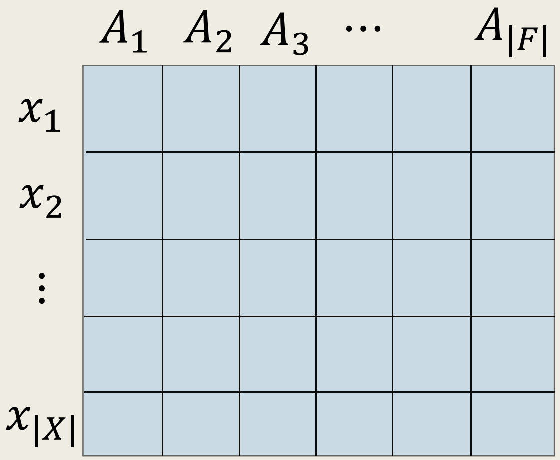
Let F be a family of subsets of some ground set X. Then

$$\sum_{x \in X} d(x) = \sum_{A \in F} |A| \ .$$

- Consider the $|X| \times |F|$ incidence matrix $M = \left(m_{x,A}\right)$, where

$$m_{x,A} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

- Then,
  - The $d(x)$ is the number of 1s in the x-th row.
  - $|A|$ is the number of 1s in the A-th column.



The matrix M

- ***The identity counts the number of 1s in the matrix M.***

- Note that, the concept of ***set family*** *is equivalent to* ***hypergraphs,*** where
  - The elements in X are the set vertices, and
  - The subsets in F are the set of hyperedges.

Average Number of Divisors

The Number of Divisors

- How many numbers from $\{1, 2, \ldots, n\}$ is a divisor of n ?
  - For any $n \geq 1$, let $t(n)$ be the number of divisors of n.
    - Ex.

      $t(p) = 2$ for any prime number p.

      $t(2^m) = m + 1$ for any integer $m \geq 1$.

Average Number of Divisors

- How many numbers from $\{1, 2, \dots, n\}$ is a divisor of n ?
  - For any $n \geq 1$, let $t(n)$ be the number of divisors of n.
  - While $t(n)$ varies a lot for different choices of n, we will see that, ***the average number of divisors***,

$$\mathrm{T}(n) := \frac{1}{n} \cdot \sum_{1 \leq i \leq n} t(i)$$

is *quite stable* and is roughly $\ln n$ for all n.

Proposition 1.10.

For any $n \geq 1$,

$$|\,\mathrm{T}(n) - \ln n\,| \leq 1\,.$$

- Consider the $n \times n$ 0-1 matrix $M = (m_{i,j})$, where

  $m_{i,j} = 1$ <u>if and only</u> i is a divisor of j.

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1		1		1		1	
3			1			1			1
4				1				1	

The number of 1s in the i-th column is exactly $t(i)$.

- Consider the $n \times n$ 0-1 matrix $M = (m_{i,j})$, where

  $m_{i,j} = 1$ if and only i is a divisor of j.

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1		1		1		1	
3			1			1			1
4				1				1	

The number of 1s in the i-th column is exactly $t(i)$.

The number of 1s in the i-th row is $\lfloor n/i \rfloor$.

- Counting the number of 1s in the matrix,

  we have $$\sum_{1 \le i \le n} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{1 \le i \le n} t(i) = n \cdot \mathrm{T}(n).$$

by the definition of $\mathrm{T}(n)$.

- We have $$\sum_{1\le i\le n} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{1\le i\le n} t(i) = n \cdot \mathrm{T}(n)\,.$$

- Since $x-1 \le \lfloor x \rfloor \le x$ holds for every real number x, we obtain
$$n \cdot \sum_{1\le i\le n} \frac{1}{i} - n \le n \cdot \mathrm{T}(n) \le n \cdot \sum_{1\le i\le n} \frac{1}{i}\,.$$

- The n^{th}-harmonic number, $H_n \coloneqq \sum_{1\le i\le n} \frac{1}{i}$, satisfies $H_n = \ln n + \gamma_n$ for some $0 \le \gamma_n \le 1$.

- Hence, we obtain $\ln n - 1 \le \mathrm{T}(n) \le \ln n + 1\,.$

Turán Number

Turán Number $T(n,k,l)$

- Consider any ground set X with n elements.

- For any $l \leq k \leq n$,

  the Turán number $\boldsymbol{T(n,k,l)}$ is *the smallest number of* l*-element subsets* of X such that

  every k*-element subset of* X *contains*

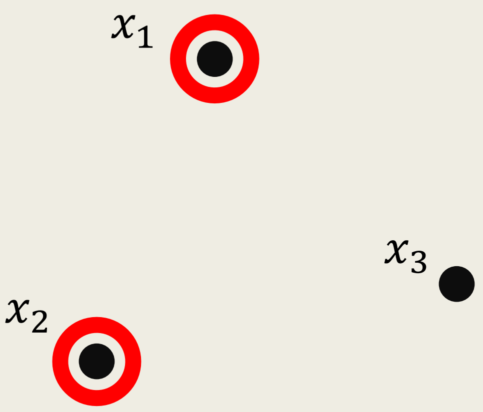
  at least one of these $\boldsymbol{l}$***-element subsets***.

How many l-element subsets do we need?

Turán Number $T(n,k,l)$

- For any $n = 3$, $k = 2$, $l = 1$, we have

$$T(3,2,1) = 2\,.$$



Any 2-element subset must contain $\{x_1\}$ or $\{x_2\}$.

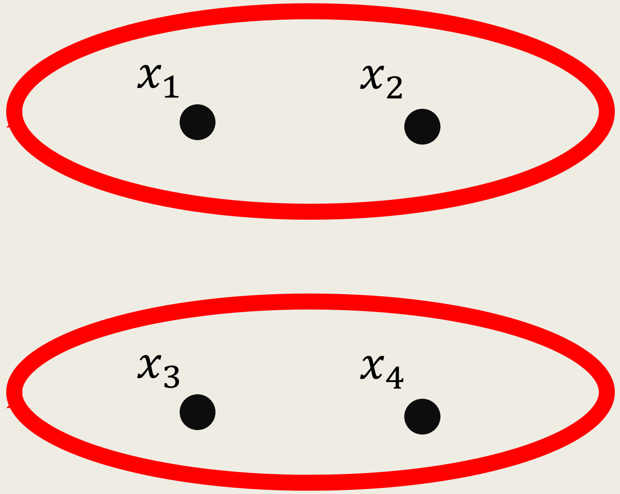
It won’t suffice, if only one 1-element subset was chosen.

One way to achieve this

Turán Number $T(n, k, l)$

- For any $n = 4$, $k = 3$, $l = 2$, we have

$$T(4,3,2) = 2\,.$$



Any 3-element subset must contain $\{x_1, x_2\}$ or $\{x_3, x_4\}$.

It won't suffice, if only one 2-element subset was chosen.

One way to achieve this

Proposition 1.9.

For all positive integers $l \leq k \leq n$,

$$T(n,k,l) \geq \binom{n}{l} \Big/ \binom{k}{l}.$$

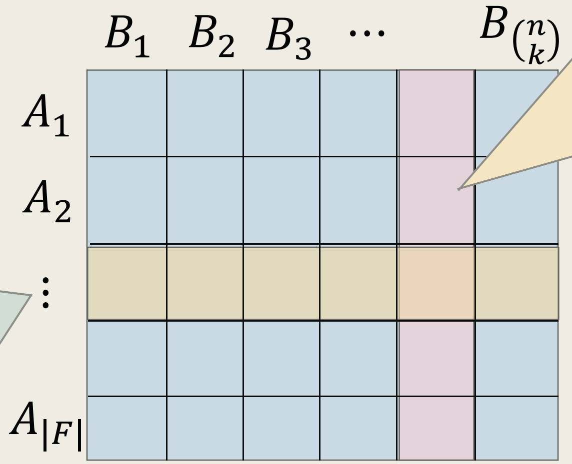
- Let F be a smallest l-uniform family over X such that every k-element subset of X contains at least one member of F.
  - In the following, we derive a lower-bound on $|F|$.

- Consider a 0-1 matrix $M = (m_{A,B})$ with size $|F| \times \binom{n}{k}$, where
  - The rows are indexed by sets A in F and
  - The columns are indexed by all possible k-element subsets of X,

and

$$m_{A,B} = \begin{cases} 1, & \text{if } A \subseteq B, \\ 0, & \text{otherwise.} \end{cases}$$

For each l-element subset A, the number of k-element subsets that contains A is exactly $\binom{n-l}{k-l}$.

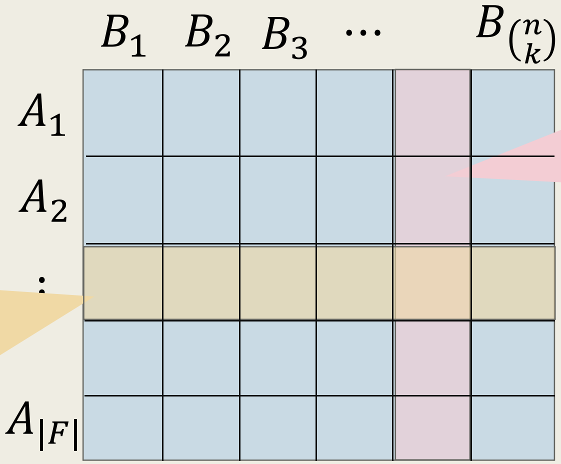


The matrix M

Since every k-element subset of X contains at least one member of F,

there is at least one 1 in each column.

For each l-element subset A,
the number of k-element subsets
containing the set A is exactly $\binom{n-l}{k-l}$.



The matrix M

Since every k-element subset of X
contains at least one member of F,
there exists at least one 1
in each column.

- Let r_A be the number of 1s in row A and c_B the number of 1s in column B.
- Counting the number of 1s in M, we have

$$|F| \cdot \binom{n-l}{k-l} = \sum_{A \in F} r_A = \sum_{B} c_B \geq 1 \cdot \binom{n}{k},$$

and

$$T(n,k,l) = |F| \geq \binom{n}{k} \Big/ \binom{n-l}{k-l} = \binom{n}{l} \Big/ \binom{k}{l}.$$

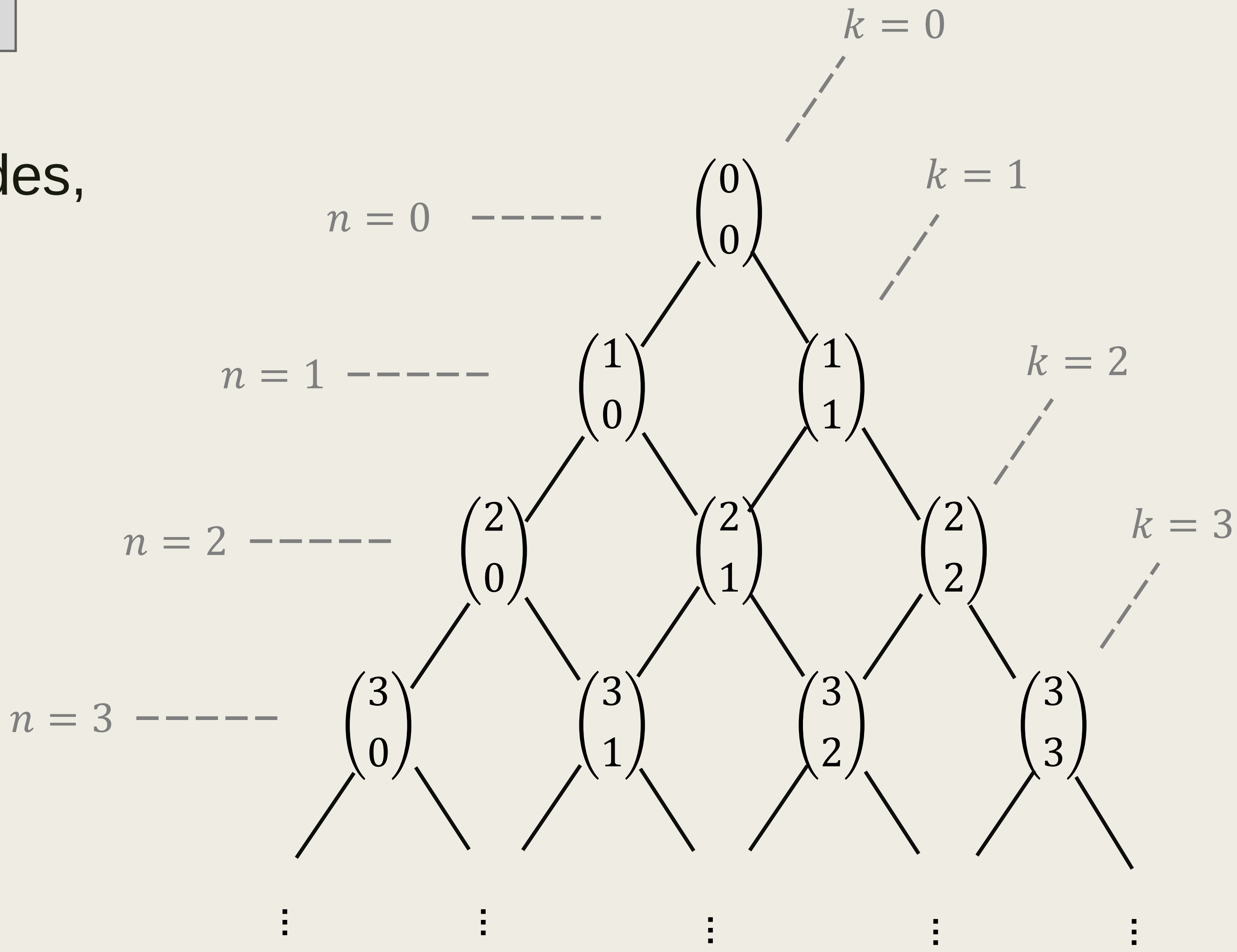
The Pascal Triangle

& Binomial Identities

The Pascal’s Triangle

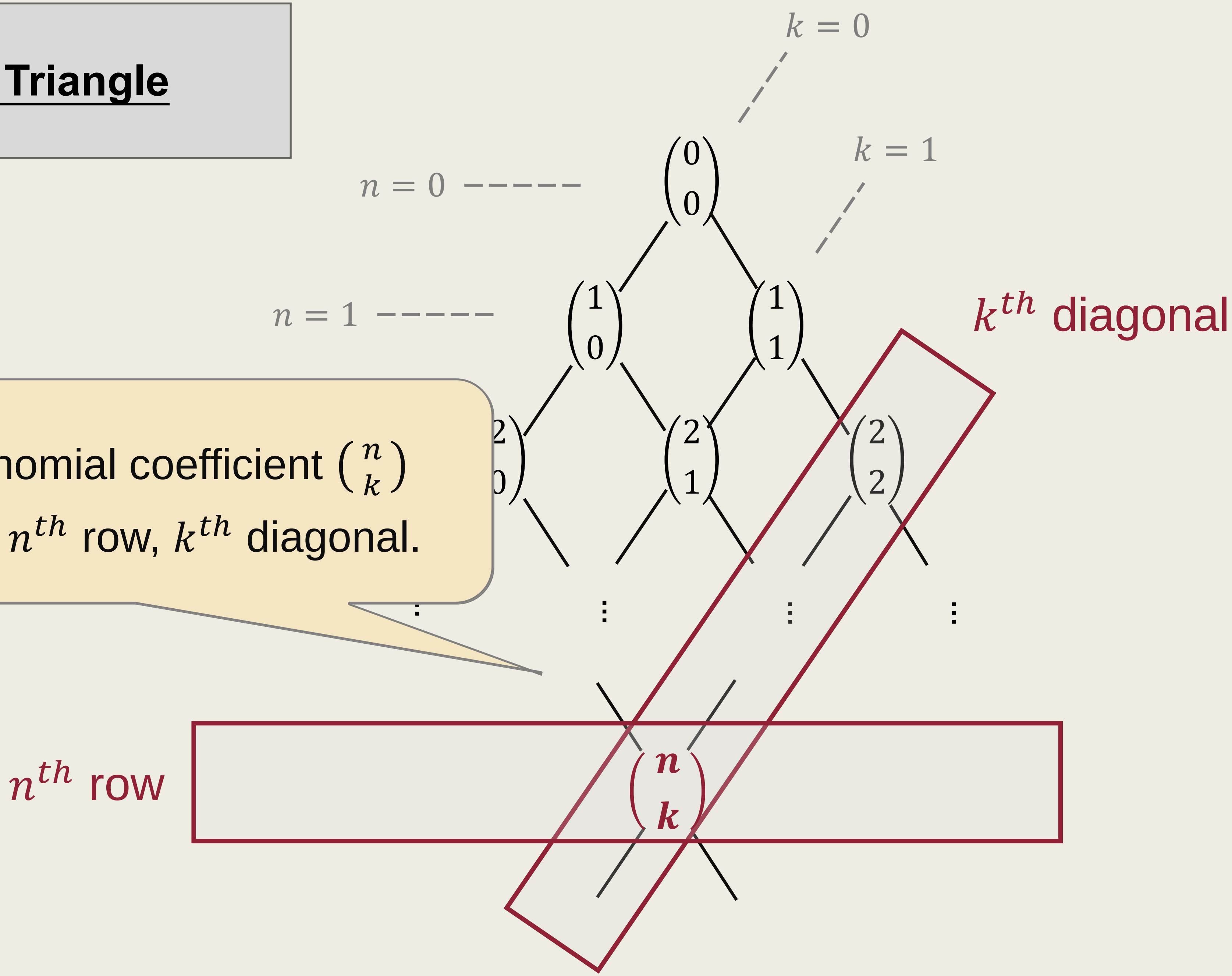
- Consider the pyramid of nodes, formed by

  the binomial coefficients.



The Pascal's Triangle

We place the binomial coefficient $\binom{n}{k}$ in the cell at the n^{th} row, k^{th} diagonal.

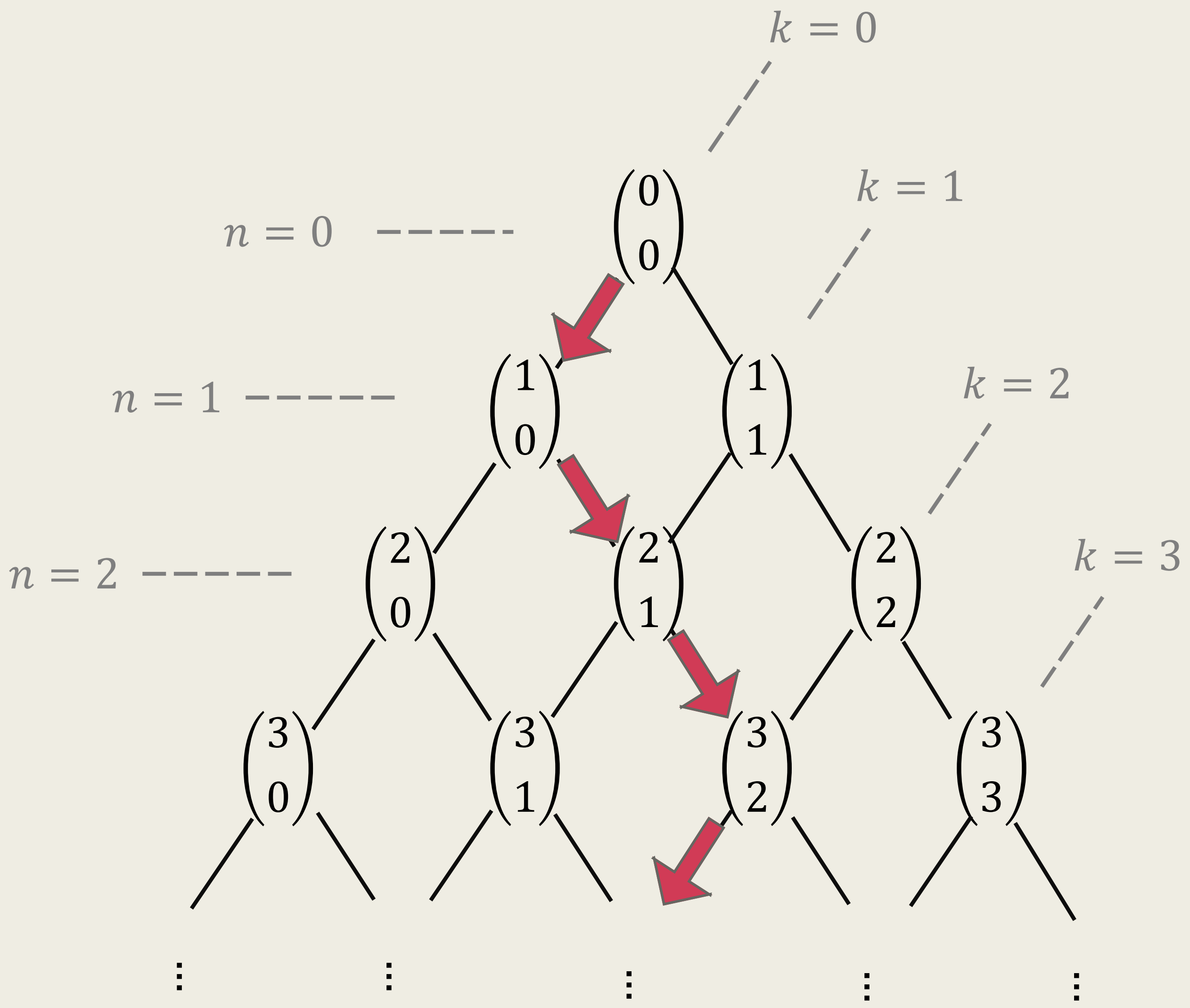


The Pascal's Triangle

- Consider any *downward path* from $(0,0)$ to (n,k).
  - Only 'L' or 'R' is allowed.

- It must use $\begin{cases} k & \text{'R's} \\ n-k & \text{'L's} \end{cases}$.

- The number of such paths is exactly $\binom{n}{k}$.

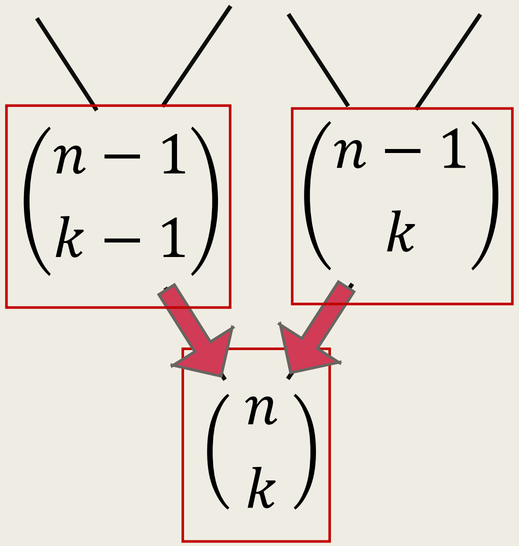


Lemma 3.

For any $n, k \in \mathbb{N}$ with $n > k$, we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- Any downward path to (n, k) must pass $(n-1, k)$ or $(n-1, k-1)$.
- The number of downward paths to (n, k) equals the sum of number of paths to $(n-1, k)$ and $(n-1, k-1)$.



<u>Lemma 4.</u>

For any $n \in \mathbb{Z}^{\geq 0}$, we have

$$\sum_{0 \leq k \leq n} \binom{n}{k} = 2^n.$$

- Consider the number of all possible *downward paths to the* n^{th} *row*.
  - It is the sum of the number of possible paths to each cell, which is $\sum_{0 \leq k \leq n} \binom{n}{k}$.
  - It is also the number of possible arrangements (permutations) with a total number of n 'L's or 'R's, and is hence 2^n.
  - By the *<u>double-counting principle</u>*, they are equal.

Lemma 5.

For any $n, r \in \mathbb{Z}^{\geq 0},\ n \geq r$,

$$\sum_{0 \leq k \leq n-r} \binom{r+k}{r} = \binom{n+1}{r+1}.$$

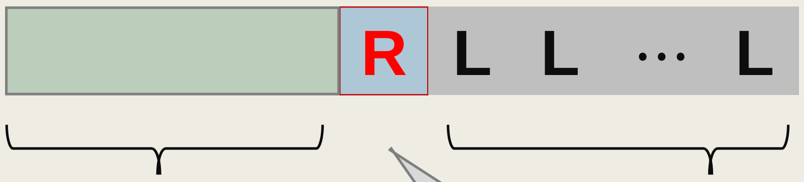
- Consider the set of all possible *downward paths to* $(n+1, r+1)$.
  - There are $\binom{n+1}{r+1}$ such paths.
  - Any of such paths must ***go to some cell at the*** r^{th}***-diagonal***, then followed by ***exactly one 'R'*** and then ***some 'L's***.

Lemma 5.

For any $n, r \in \mathbb{Z}^{\geq 0}$, $n \geq r$,

$$\sum_{0 \leq k \leq n-r} \binom{r+k}{r} = \binom{n+1}{r+1}.$$

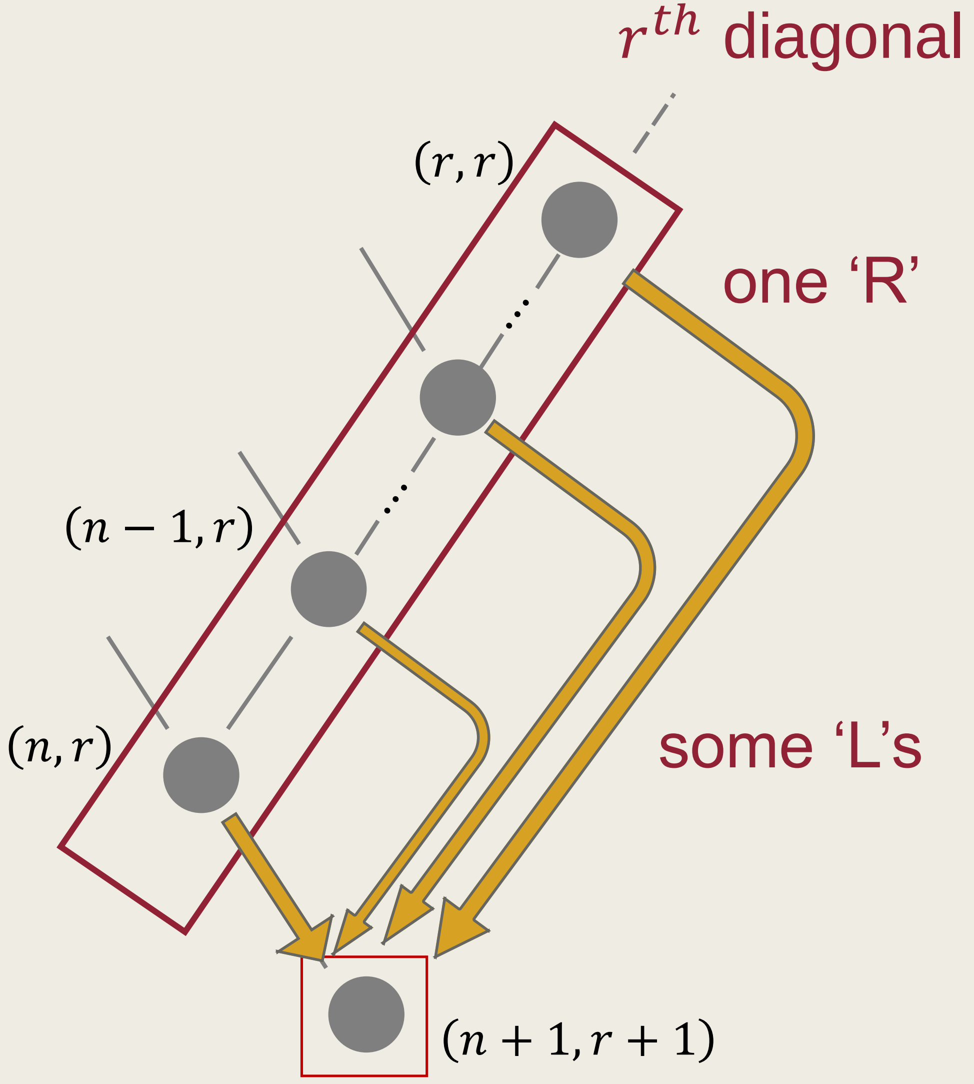
A downward path to $(n+1, r+1)$



A downward path to (ℓ, r) for some $r \leq \ell \leq n$

Zero or more 'L's

The last 'R' in the path.



- Consider the set of all possible *downward paths to* $(n+1, r+1)$.
  - There are $\binom{n+1}{r+1}$ such paths.
  - Any of such paths must ***go to some cell at the r^{th}-diagonal***, then followed by ***exactly one 'R'*** and then ***some 'L's***.
    - Identify such paths by its last 'R'.
    - Then, there are $\sum_{r \le \ell \le n} \binom{\ell}{r} = \sum_{0 \le k \le n-r} \binom{r+k}{r}$ such paths.
  - By double counting principle, they are equal.

Lemma 6.

For any $n \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq n} \binom{n}{k}^2 = \binom{2n}{n}.$$

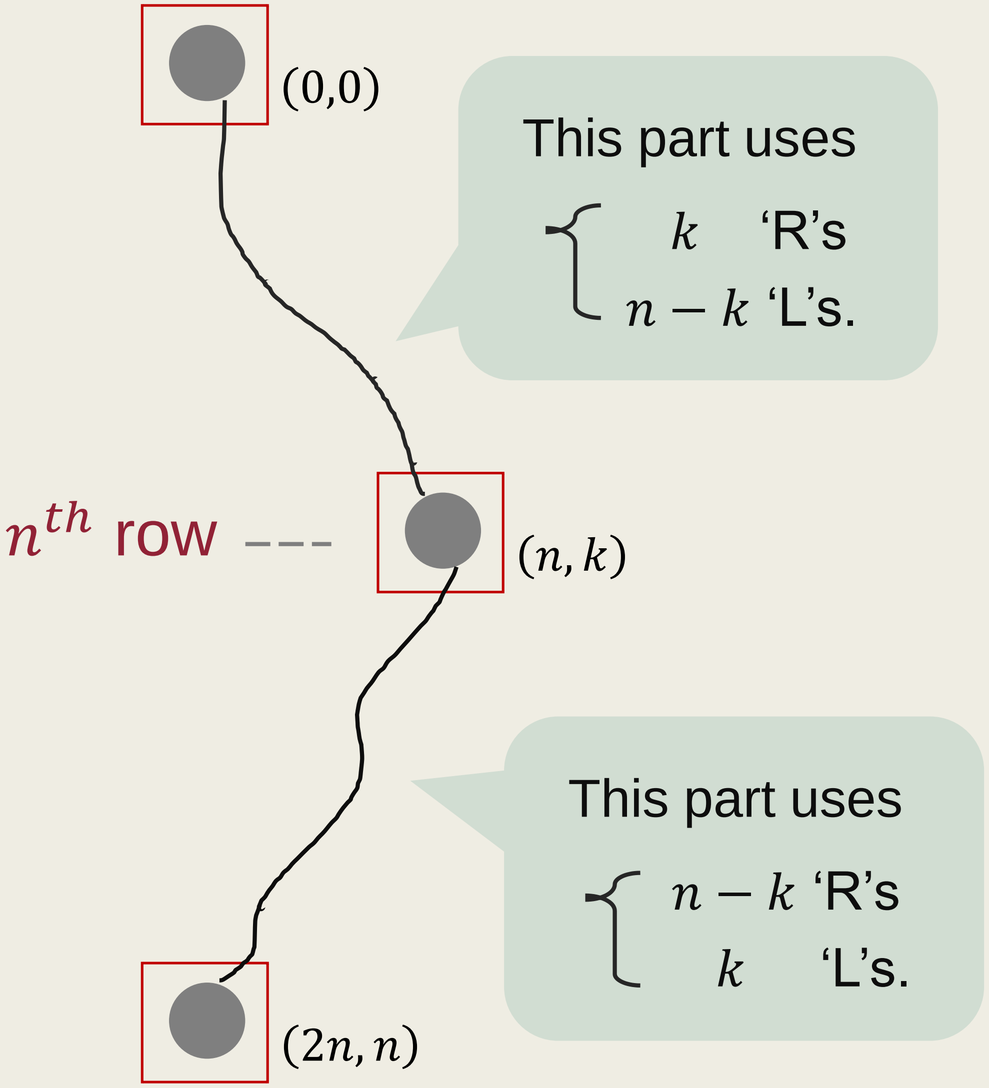
- Consider the set of all possible *downward paths to* $(2n, n)$.
  - There are $\binom{n+1}{r+1}$ such paths.
  - Identify any of such paths by the cell it reaches at the n^{th}-row.
    - Suppose that it is (n, k).
    - Let's count the number of such paths.

Lemma 6.

For any $n \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq n} \binom{n}{k}^2 = \binom{2n}{n}.$$

- The upper-part uses k ‘R’s and $n-k$ ‘L’s.
  - There are $\binom{n}{k}$ such paths.
- The lower-part uses $n-k$ ‘R’s and k ‘L’s.
  - There are $\binom{n}{n-k} = \binom{n}{k}$ such paths.



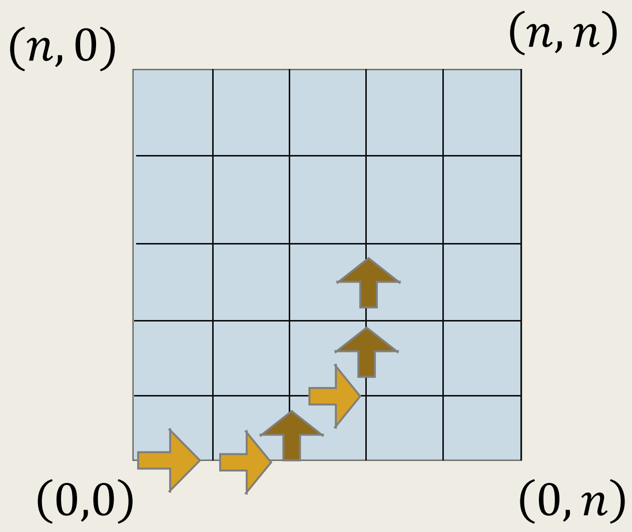
- Consider the set of all possible *downward paths to* $(2n, n)$.
  - There are $\binom{n+1}{r+1}$ such paths.
  - Identify any of such paths by the cell it reaches at the n^{th}-row.
    - Suppose that it is (n, k).
    - By the above argument, there are $\binom{n}{k}^2$ such paths.
  - Taking summation over the cells at the n^{th}-row,

    there are $\sum_{0 \leq k \leq n} \binom{n}{k}^2$ such paths.
- By the double-counting principle, they are equal.

The Catalan Numbers

The Catalan Number C_n

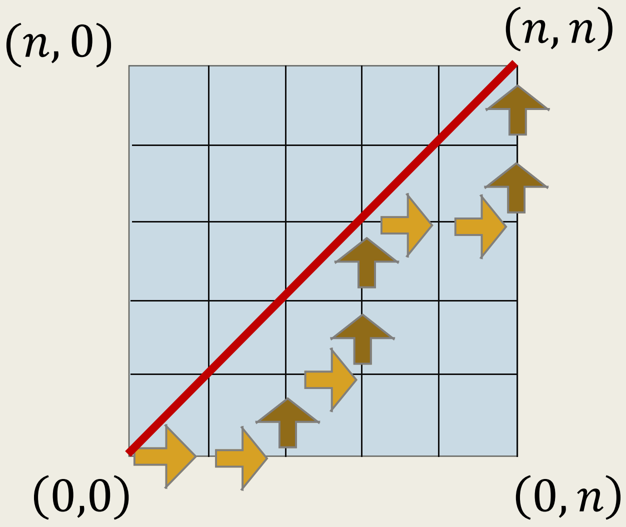
- Consider the $n \times n$ grid points and any path from $(0,0)$ to (n, n) that uses only 'up' and 'right'.



The Catalan Number C_n

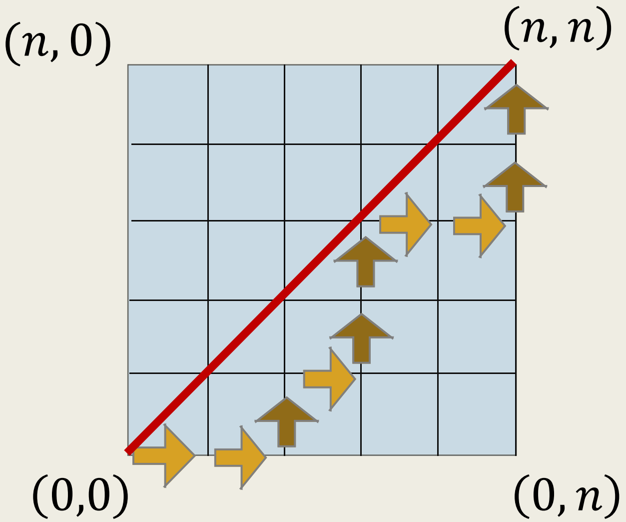
- Consider the $n \times n$ grid points and any path from $(0,0)$ to (n,n) that uses only ‘up’ and ‘right’.
- Define the Catalan number C_n to be the number of possible paths that never cross the diagonal connecting $(0,0)$ and (n,n).
- We will prove that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$



- We will prove that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$



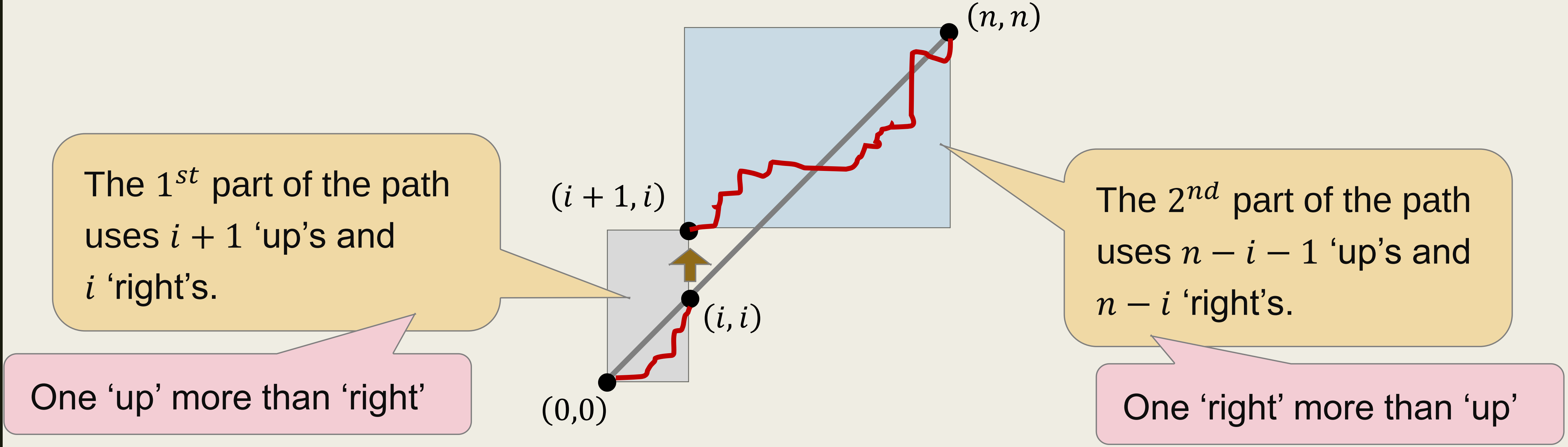
- It is clear that without extra restrictions,

  the set of all possible paths that go from $(0,0)$ to (n,n) is $\binom{2n}{n}$.

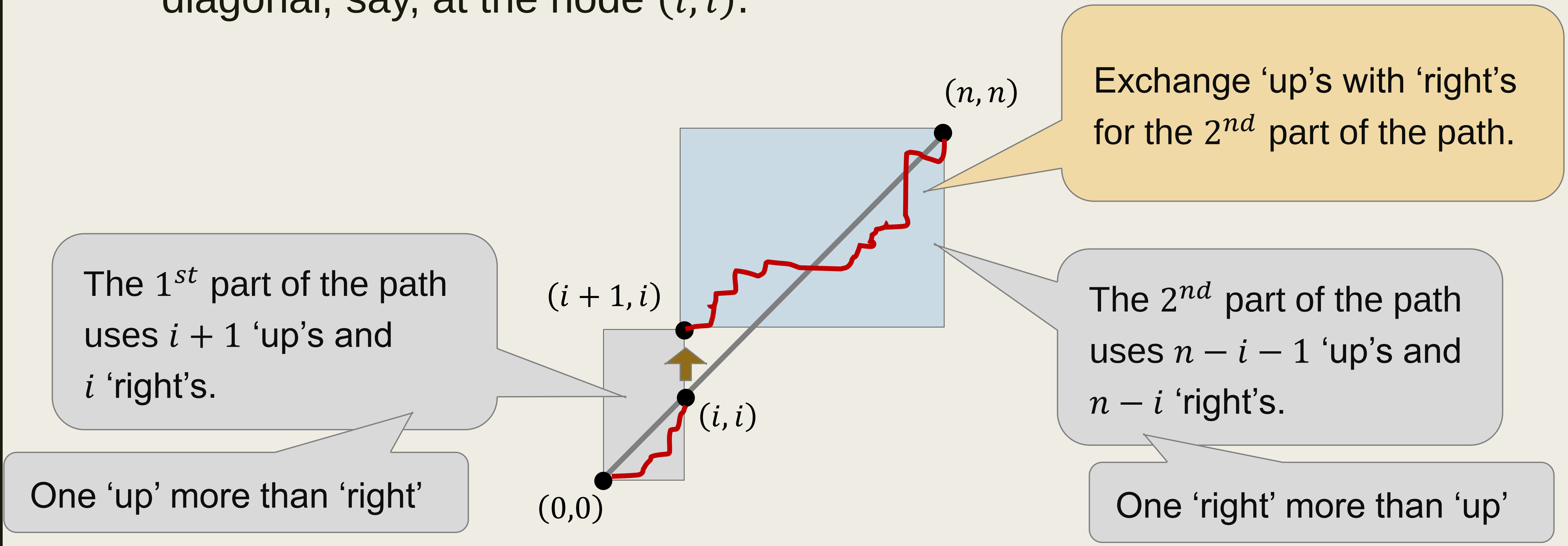
- It suffices to prove that,

  the number of paths *that have crossed the diagonal* is exactly $\binom{2n}{n-1}$.

- It suffices to prove that,
  the number of paths *that have crossed the diagonal* is $\binom{2n}{n-1}$.
- Consider any of such path and the first time that it crosses the diagonal, say, at the node (i, i).



- It suffices to prove that, the number of paths *that have crossed the diagonal* is $\binom{2n}{n-1}$.
- Consider any of such path and the first time that it crosses the diagonal, say, at the node (i, i).



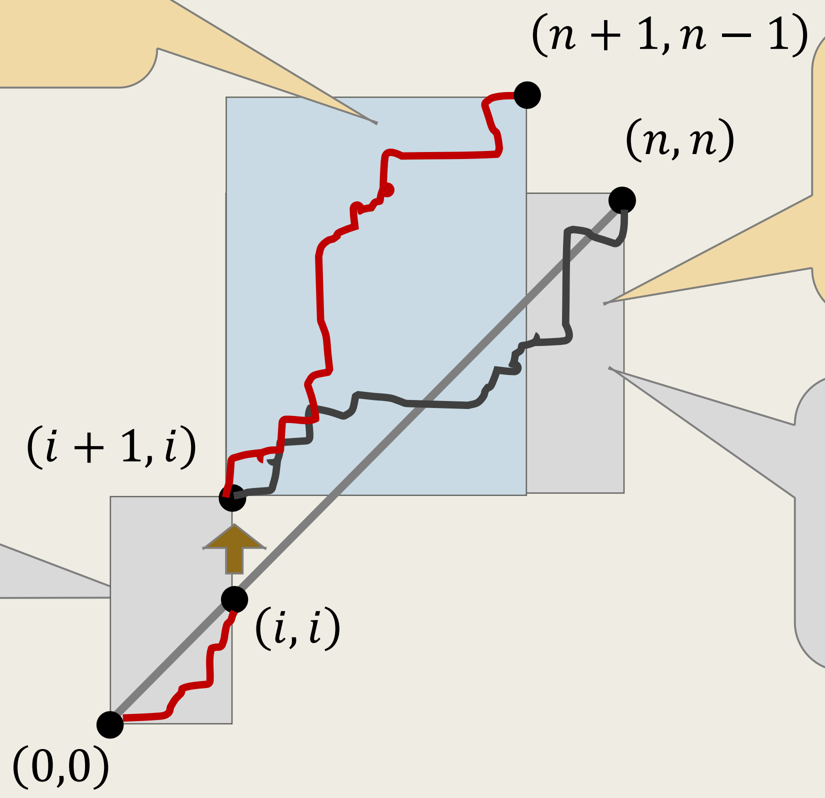
- Consider any of such path and the first time that it crosses the diagonal, say, at the node (i, i).

Hence, it will reach $(n+1, n-1)$.

The new path has $n-i$ 'up's and $n-i-1$ 'right's.

One 'up' more than 'right'

Exchange 'up's with 'right's for the 2^{nd} part of the path.



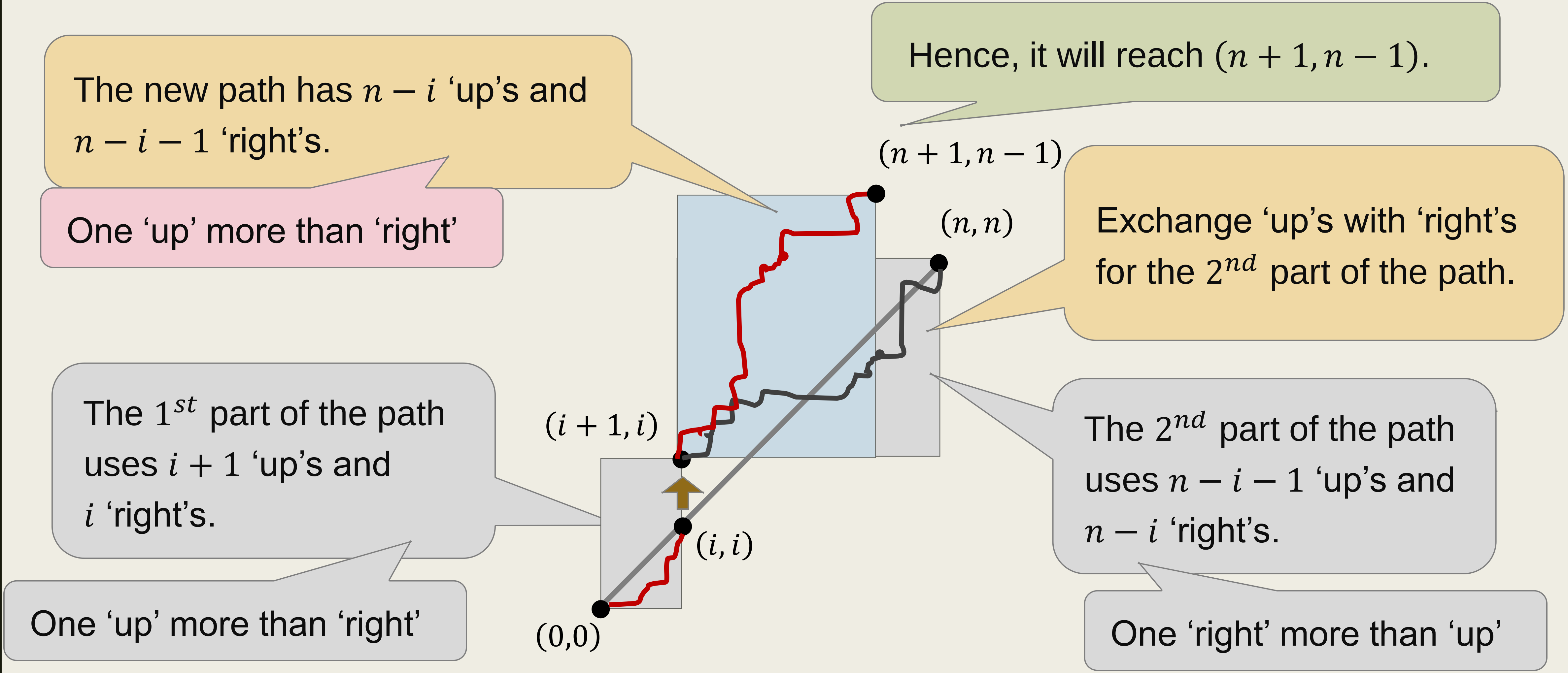
The 1^{st} part of the path uses $i+1$ 'up's and i 'right's.

One 'up' more than 'right'

The 2^{nd} part of the path uses $n-i-1$ 'up's and $n-i$ 'right's.

One 'right' more than 'up'

- Observe that, this is **a one-to-one correspondence** between *paths that cross the diagonal* and *paths that reach* $(n+1, n-1)$.



- It suffices to prove that,

  the number of paths *that have crossed the diagonal* is $\binom{2n}{n-1}$.

- Consider any of such path and the first time that it crosses the diagonal, say, at the node (i, i).

  - There is *a one-to-one correspondence* between *paths that cross the diagonal and paths that reach* $(n+1, n-1)$.

  - Hence,

    the number of paths that have crossed the diagonal is $\binom{2n}{n-1}$.

The Density of 0-1 Matrices

- Let H be an $m \times n$ 0-1 matrix and $0 \leq \alpha \leq 1$ be a real number.
  - We say that H is ***α-dense***,
    if *at least an α-fraction of its entries* are 1s,
    i.e.,
    $$\#\text{ of 1s in } H \geq \alpha \cdot mn\,.$$
  - Similarly, a row (column) is ***α-dense***,
    if at least an α-fraction of its entries are 1s.

Lemma 2.13 (Grigni and Sipser 1995).

Note that, $\sqrt{\alpha} \geq \alpha$ when $\alpha \leq 1$.

If H is 2α-dense, then either

1. There exists a row which is $\sqrt{\alpha}$-dense, or
2. At least $\sqrt{\alpha} \cdot m$ of the rows are α-dense.

- If H is dense, then either
  - There exists a row that is very dense, or
  - A certain fraction of rows must be dense enough.
- Intuitively,
  if the first condition does not hold, then the second must hold.

<u>Lemma 2.13 (Grigni and Sipser 1995).</u>

If H is 2α-dense, then either

1. There exists a row which is $\sqrt{\alpha}$-dense, or
2. At least $\sqrt{\alpha} \cdot m$ of the rows are α-dense.

- Suppose that both of the cases do not hold.
  - By 2, less than $\sqrt{\alpha} \cdot m$ rows are α-dense.
  - By 1, each of the above rows has less than $\sqrt{\alpha} \cdot n$ 1s.
  - Hence, the total number of 1s in ***these α-dense rows*** is less than $\sqrt{\alpha} \cdot \sqrt{\alpha} \cdot mn = \alpha \cdot mn$.

- Suppose that both of the cases do not hold.
  - By 2, less than $\sqrt{\alpha} \cdot m$ rows are **α-dense**.
  - By 1, each of the above rows has less than $\sqrt{\alpha} \cdot n$ 1s.
  - Hence, the total number of 1s in ***these α-dense rows*** is less than $\sqrt{\alpha} \cdot \sqrt{\alpha} \cdot mn = \alpha \cdot mn$.

- Consider the remaining ***non-α-dense rows***.
  - The number of 1s in these rows is less than $\alpha \cdot mn$.

- Hence, the total number of 1s in H is strictly less than $2\alpha \cdot mn$, a contradiction.

The Number of Sufficiently Dense Rows (Columns) in a Dense Matrix.

Q: How many rows or columns of an α-dense matrix will be "***dense enough***?"

- In the following,
  let's use a <u>*general setting*</u> to answer this question!

- Let $A_1, A_2, \ldots, A_k$ be finite sets, and
  consider the Cartesian product of A_i

$$A = A_1 \times A_2 \times \cdots \times A_k \ .$$

Then, the Cartesian product
$A = \{\ (i,j) : 1 \le i \le m, 1 \le j \le n\ \}$
is the <u>*coordinates of the entries*</u>.

<u>*To relate the two concepts*</u>, for $m \times n$ 0-1 matrix,
we have $A_1 = \{1,2,\ldots,m\}$, $A_2 = \{1,2,\ldots,n\}$.

- Let $A_1, A_2, \ldots, A_k$ be finite sets, and consider the Cartesian product of A_i

$$A = A_1 \times A_2 \times \cdots \times A_k .$$

To relate the two concepts, for an $m \times n$ 0-1 matrix, we have $A_1 = \{1,2,\ldots,m\}$, $A_2 = \{1,2,\ldots,n\}$.

- Let $D \subseteq A$ be the set of *elements of interests*.

The set of entries that are 1.

  - For any $b \in A_i$, define the ***degree of b*** in D as

number of 1s in the b^{th}-row (column).

$$d_D(b) \coloneqq |\{a \in D \colon a_i = b\}| ,$$

i.e., the number of elements of interests whose i^{th}-coordinate is b.

- We say that a point $b \in A_i$ is popular in D, if

$$d_D(b) \geq \frac{1}{2k} \cdot \frac{|D|}{|A_i|},$$

i.e., the degree of b is at least $1/2k$ times the average degree of the elements in A_i.

number of 1s (or, density) in that row (column) is at least $1/4$ of the average.

- Let $P_i \subseteq A_i$ be the set of all popular elements in A_i, and let

$$P \coloneqq P_1 \times P_2 \times \cdots \times P_k .$$

The entries formed by popular rows and columns.

Lemma 2.14 (Håstad).

$|P| > |D|/2.$

Lemma 2.14 (Håstad).

$|P| > |D|/2.$

Since

$$|P| \geq |P \cap D| = |D| - |D \setminus P|$$

- It suffices to prove that $|D \setminus P| < |D|/2$.
  - For every non-popular point $b \in A_i$, we have

$$d_D(b) = |\{a \in D: a_i = b\}| < \frac{1}{2k} \cdot \frac{|D|}{|A_i|}.$$

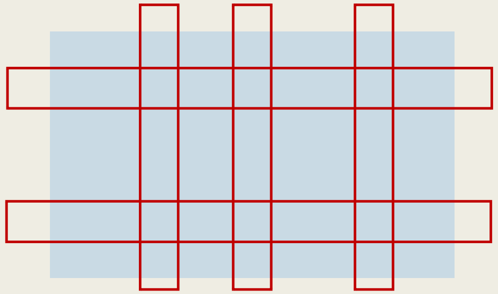
  - Counting the elements in $|D \setminus P|$, we have

$$|D \setminus P| \leq \sum_{1 \leq i \leq k} \sum_{b \in A_i \setminus P_i} d_D(b) < \sum_{1 \leq i \leq k} \sum_{b \in A_i \setminus P_i} \frac{1}{2k} \cdot \frac{|D|}{|A_i|}$$

$$\leq \sum_{1 \leq i \leq k} \frac{1}{2k} \cdot |D| = \frac{1}{2}|D|.$$

Any element in $|D \setminus P|$ is counted at least once in the summation.

$|A_i \setminus P_i| \leq |A_i|$



Q: How many rows or columns of an α-dense matrix will be "***dense enough***?"

- Interpret D as the entries with 1 in the matrix M.
  - Lemma 2.14 says that,
    the size of $|P| = |P_1| \cdot |P_2|$ is lower-bounded by $|D|/2$.
  - Provided that $|D|$ is large, at least one of $|P_1|$, $|P_2|$ must be large.

Corollary 2.15.

In any 2α-dense 0-1 matrix H, either a $\sqrt{\alpha}$-fraction of its rows or a $\sqrt{\alpha}$-fraction of its columns (or both) are $\alpha/2$-dense.

Corollary 2.15.

In any 2α-dense 0-1 matrix H, either a $\sqrt{\alpha}$-fraction of its rows or a $\sqrt{\alpha}$-fraction of its columns (or both) are $\alpha/2$-dense.

- Let H be an $m \times n$ matrix and D be the set of entries of 1.
- Let P_1 and P_2 be the set of popular rows and columns, respectively.
  - For any $i \in P_1$, we have

$$\deg_D(i) \geq \frac{1}{2k} \cdot \frac{|D|}{|A_1|} \geq \frac{1}{4} \cdot \frac{2\alpha \cdot mn}{m} \geq \frac{\alpha}{2} \cdot n \,,$$

    which implies that i is $\alpha/2$-dense.
  - Similarly, any $j \in P_2$ is also $\alpha/2$-dense.

- Let H be an $m \times n$ matrix and D be the set of entries of 1.
- Let P_1 and P_2 be the set of popular rows and columns, respectively.
  - Any row / column $b \in P_1 \cup P_2$ is $\alpha/2$-dense.

- By Lemma 2.14, $|P_1| \cdot |P_2| \geq |D| / 2 \geq \alpha \cdot mn$,

  which implies that $\dfrac{|P_1|}{m} \cdot \dfrac{|P_2|}{n} \geq \alpha$.

- Hence, either $\dfrac{|P_1|}{m} \geq \sqrt{\alpha}$ or $\dfrac{|P_2|}{n} \geq \sqrt{\alpha}$ must hold.
- This proves the statement of this corollary.

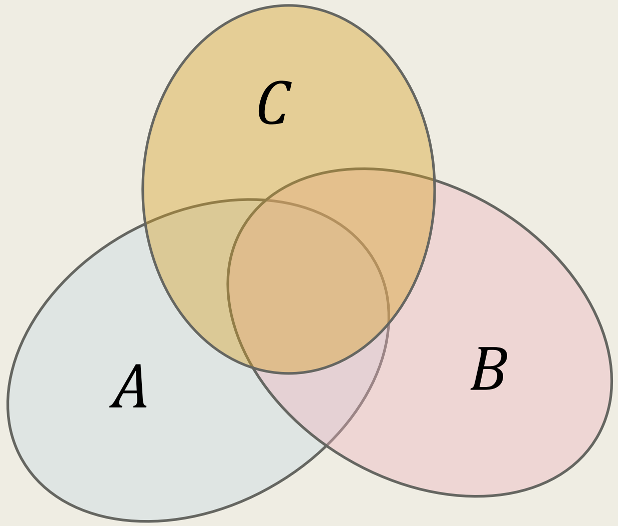
Principle of Inclusion and Exclusion

The Inclusion-Exclusion Principle

- Let $A_1, A_2, \dots, A_n \subseteq X$ be given sets.

  For any $I \subseteq \{1,2,\dots,n\}$,

  define $A_I \coloneqq \bigcap_{i \in I} A_i$ with the convention that $A_\phi = X$.



- Let $A_1, A_2, \ldots, A_n \subseteq X$ be given sets. For any $I \subseteq \{1,2,\ldots,n\}$, define $A_I := \bigcap_{i \in I} A_i$ with the convention that $A_\phi = X$.

Theorem 3. (The inclusion-exclusion principle)

Let $A_1, A_2, \ldots, A_n$ be a sequence of sets. We have

$$\left| \bigcup_{1 \le i \le n} A_i \right| = \sum_{\substack{I \subseteq \{1,2,\ldots,n\}, \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot |A_I|$$

$$= \sum_{0 < k \le n} \sum_{\substack{I \subseteq \{1,2,\ldots,n\}, \\ |I| = k}} (-1)^{k+1} \cdot |A_I| .$$

Let's first derive $\left|\bigcap_{1 \le i \le n} \overline{A_i}\right|$.

Proposition 1.13 (Inclusion-Exclusion Principle).

Let $A_1, \dots, A_n$ be subsets of X.
Then the number of elements of X which lie in none of the subsets A_i is

$$\sum_{I \subseteq \{1,2,\dots,n\}} (-1)^{|I|} \cdot |A_I| \ .$$

- Rewrite the sum as

$$\sum_{I} (-1)^{|I|} \cdot |A_I| = \sum_{I} \sum_{x \in A_I} (-1)^{|I|} = \sum_{x} \sum_{I : x \in A_I} (-1)^{|I|} \ .$$

- For each $x \in X$, consider the contribution of x to the above summation.
  - If $x \notin A_i$ for all i, then the only term in the sum to which x contributes is $I = \emptyset$, and the contribution is 1.

- Rewrite the sum as

$$\sum_I (-1)^{|I|} \cdot |A_I| = \sum_I \sum_{x \in A_I} (-1)^{|I|} = \sum_x \sum_{I : x \in A_I} (-1)^{|I|} .$$

- For each $x \in X$, consider the contribution of x to the above summation.
  - If $x \in A_i$ for all i, define

    $$J = \{ i : x \in A_i \} \neq \emptyset .$$

    Then $x \in A_I$ if and only if $I \subseteq J$.
  - Thus, the contribution is

    $$\sum_{I \subseteq J} (-1)^{|I|} = \sum_{0 \le i \le |J|} \binom{|J|}{i} \cdot (-1)^i = (1-1)^{|J|} = 0 .$$

  - So, the overall sum is the number of points lying in none of the sets.

Proposition 1.14 (Inclusion-Exclusion Principle).

Let $A_1, \ldots, A_n$ be subsets of X. Then

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \cdot |A_I| \; .$$

- We have $|A_1 \cup \cdots \cup A_n| = \left|A_\emptyset\right| - \left|\overline{A_1} \cap \cdots \cap \overline{A_n}\right|$.
- By Proposition 1.13, we obtain

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \cdot |A_I| \; .$$