There are 4 problems, accounting for 100% in total.

Problem 1 (25%)**.** Given an undirected graph $G = (V, E)$ with edge weight function $w\colon E \to \mathbb{Q}^+$ and a set of k terminals $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$, the multiway cut problem is to compute a set of edges A with minimum weight such that the removal of A will disconnect the terminals in S from each other.

For each $1 \le i \le k$, define the isolating cut for v_i to be a set of edges whose removal disconnects v_i from other terminals in S.

Consider the following greedy algorithm for multiway cut.

1. For $1 \le i \le k$, compute the minimum-cost isolating cut for v_i, by the max-flow min-cut algorithm. Let the cut be A_i.
2. Output $\bigcup_{1 \le i \le k} A_i$.

Show that the above algorithm computes a 2-approximation for the multiway cut problem. Describe how the algorithm can be modified to compute a $2(1-1/k)$-approximation. Justify the approximation guarantee.

Problem 2 (25%)**.** Let $G = (V, E)$ be a graph. Consider the following algorithm for the vertex cover problem.

Algorithm 1 Greedy-Algo-4-VC

```
U ⟵ ∅,   E' ⟵ E.
while E' is not empty do
    Pick an arbitrary e = (u, v) from E'.
    Add u and v to U.
    Remove all incident edges of u and v from E'.
end while
Output U.
```

1. Prove that the above algorithm produces a 2-approximation for vertex cover on G.
2. Consider the following dual LP for the natural LP of the vertex cover problem.

$$
\begin{aligned}
\max \quad & \sum_{e \in E} y_e & & \text{LP-(*)} \\
\text{s.t.} \quad & \sum_{e \in E\colon e \ni v} y_e \le 1, & & \forall v \in V, \\
& y_e \ge 0, & & \forall e \in E.
\end{aligned}
$$

   Interpret the above greedy algorithm as a dual-fitting process for LP-(*). Justify your answer.

Problem 3 (25%)**.** Recall that, in the separation problem for a linear polytope $Q \subseteq \mathbb{R}^n$, we are given a point $q \in \mathbb{R}^n$, and the goal is either to confirm that $q \in Q$ or to produce a separating hyperplane for q and Q.

Let $G = (V, E)$ be an undirected graph with edge weight function $w \colon E \to \mathbb{Q}^+$ and $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$ be a set of k terminals. For any $A \subseteq V$, define $\delta(A)$ to be the set of edges running between A and $V \setminus A$, i.e., the cut edges of A and $V \setminus A$.

For any $A \subseteq V$, define the connectivity requirement function $f(A)$ as follows.

$$f(A) := \begin{cases} 1, & \text{if } \emptyset \neq A \cap S \subsetneq S, \\ 0, & \text{otherwise.} \end{cases}$$

Consider the following LP relaxation for the Steiner tree problem.

$$\begin{array}{llll} \min & \sum_{e \in E} w_e \cdot x_e & & \text{LP-(*)} \\ \text{s.t.} & \sum_{e \in \delta(A)} x_e \geq f(A), & \forall A \subseteq V, & \\ & x_e \geq 0, & \forall e \in E. & \end{array}$$

Derive a separation oracle that can be used to answer the separation problem for this LP. Justify the correctness of your algorithm.

Problem 4 (25%)**.** In the partial vertex cover (PVC) problem, we are given a graph $G = (V, E)$ and a nonnegative integer $k \in \mathbb{Z}^{\geq 0}$, and the goal is to compute a minimum size subset $U \subseteq V$ that covers at least k edges. Define $\ell := |E| - k$. We have the following natural LP.

$$\begin{array}{llll} \min & \sum_{v \in V} x_v & & \text{LP-(PVC)} \\ \text{s.t.} & x_u + x_v + p_e \geq 1, & \forall e = (u, v) \in E, & \\ & \sum_{e \in E} p_e \leq \ell, & & \\ & x_v,\ p_e \geq 0, & \forall v \in V,\ e \in E. & \end{array}$$

1. Prove that, in any extreme point (basic feasible) solution $(\boldsymbol{x}, \boldsymbol{p})$ for LP-(PVC), at least one of the following two conditions must hold.
   (a) $x_v = 0$ for some $v \in V$ or $p_e \in \{0, 1\}$ for some $e \in E$.
   (b) $x_v \geq 1/2$ holds for some $v \in V$.
2. Based on the above property, design an iterative rounding algorithm that produces a 2-approximation for the PVC problem. Justify your answer.