Introduction to
Approximation Algorithms

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Outline

- Introduction to Linear Programming Duality
  - The Weak Duality Theorem
  - Deriving the Dual LP
  - Examples of Natural Primal and Dual Problems
- 2-Approximation for Vertex Cover via the Dual-Fitting Technique

LP Duality

Dual LP as a systematic process to bounding the primal LP.

Deriving Bounds for the LP

- Consider the following minimization LP.

$$\begin{aligned} \min \quad & 7x_1 + x_2 + 5x_3 \qquad (*) \\ \text{s.t.} \quad & x_1 - x_2 + 3x_3 \geq 10, \\ & 5x_1 + 2x_2 - x_3 \geq 6, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Let's denote the objective by $\boldsymbol{Obj}$.

- How can we derive a lower-bound on Obj (and hence OPT$(*)$) ?

How small can the value of this LP be?

Lower-Bound Ver. 1

$$
\begin{aligned}
\min \quad & 7x_1 + x_2 + 5x_3 && (*) \\
\text{s.t.} \quad & x_1 - x_2 + 3x_3 \geq 10, \\
& 5x_1 + 2x_2 - x_3 \geq 6, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

(1)

Let's denote the objective by $\boldsymbol{Obj}$.

- How can we derive a lower-bound on Obj (and hence OPT$(*)$) ?

Since $\boldsymbol{x_1, x_2, x_3 \geq 0}$, we have

$$Obj = 7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10.$$

$\mathbf{7}x_1 \geq x_1$

$x_2 \geq -x_2$

$\mathbf{5}x_3 \geq \mathbf{3}x_3$

By (1)

We get a lower-bound of **10**.

How small can the value of this LP be?

Lower-Bound Ver. 2

$$\min \quad 7x_1 + x_2 + 5x_3 \qquad (*)$$

$$\text{s.t.} \quad x_1 - x_2 + 3x_3 \geq 10,$$

$$5x_1 + 2x_2 - x_3 \geq 6,$$

$$x_1, x_2, x_3 \geq 0.$$

(1)

(2)

Let's denote the objective by $\boldsymbol{Obj}$.

- How can we derive a lower-bound on Obj (and hence OPT$(*)$) ?

Since $\boldsymbol{x_1, x_2, x_3 \geq 0}$, we have

$$Obj = 7x_1 + x_2 + 5x_3$$

$$\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \geq 10 + 6 = 16.$$

$\mathbf{7}x_1 \geq \mathbf{6}x_1$

$x_2 \geq x_2$

$\mathbf{5}x_3 \geq \mathbf{2}x_3$

By (1) + (2)

We get a lower-bound of **16**.

How small can the value of this LP be?

What is the best value we can get ? We ask.

Lower-Bound Ver. 3

$$\min \quad 7x_1 + x_2 + 5x_3 \qquad (*)$$

$$\text{s.t.} \quad x_1 - x_2 + 3x_3 \geq 10, \qquad (1)$$

$$5x_1 + 2x_2 - x_3 \geq 6, \qquad (2)$$

$$x_1, x_2, x_3 \geq 0.$$

Let's denote the objective by $\boldsymbol{Obj}$.

- How can we derive a lower-bound on Obj (and hence OPT$(*)$) ?

Since $\boldsymbol{x_1, x_2, x_3 \geq 0}$, we have

$$Obj = 7x_1 + x_2 + 5x_3$$

$$\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \geq 20 + 6 = 26.$$

$\mathbf{7}x_1 \geq \mathbf{7}x_1$

$x_2 \geq \mathbf{0}x_2$

$\mathbf{5}x_3 \geq \mathbf{5}x_3$

By 2*(1) + (2)

This time,

we get **26**!

How small can the value of this LP be?

Obtaining the Best Lower-Bound for $(*)$

$$\min \quad 7x_1 + x_2 + 5x_3 \qquad (*)$$

$$\text{s.t.} \quad x_1 - x_2 + 3x_3 \geq 10,$$

$$5x_1 + 2x_2 - x_3 \geq 6,$$

$$x_1, x_2, x_3 \geq 0.$$

Multiply by $y_1 \geq 0$

Multiply by $y_2 \geq 0$

$$\max \quad 10y_1 + 6y_2 \qquad (**)$$

$$\text{s.t.} \quad y_1 + 5y_2 \leq 7,$$

$$-y_1 + 2y_2 \leq 1,$$

$$3y_1 - y_2 \leq 5,$$

$$y_1, y_2 \geq 0.$$

We get a new LP $(**)$.

Any feasible solution for $(**)$ gives a valid lower-bound on the value of $(*)$!

$(1,0) \rightarrow 10$

$(1,1) \rightarrow 16$

$(2,1) \rightarrow 26$

We want to maximize the lower-bound obtained!

The combined coefficient cannot exceed the coefficient of Obj.

Obtaining the Best Upper-Bound for $(**)$

$$
\begin{aligned}
\max\quad & 10y_1 + 6y_2 \qquad\qquad (**)\\
\text{s.t.}\quad & y_1 + 5y_2 \le 7,\\
& -y_1 + 2y_2 \le 1,\\
& 3y_1 - y_2 \le 5,\\
& y_1, y_2 \ge 0.
\end{aligned}
$$

Multiply by $x_1 \ge 0$

Multiply by $x_2 \ge 0$

Multiply by $x_3 \ge 0$

$$
\begin{aligned}
\min\quad & 7x_1 + x_2 + 5x_3 \qquad\qquad (*)\\
\text{s.t.}\quad & x_1 - x_2 + 3x_3 \ge 10,\\
& 5x_1 + 2x_2 - x_3 \ge 6,\\
& x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

We want to minimize the upper-bound obtained!

The combined coefficient must be at least the coefficient of Obj.

Apply the same idea on $(**)$ and we get the LP $(*)$!

Moreover, the two LPs have the same optimal value.

$$(x_1, x_2, x_3) = \left(\frac{7}{4}, 0, \frac{11}{4}\right) \rightarrow 26$$

$$(y_1, y_2) = (2,1) \rightarrow 26$$

Primal and Dual LPs

- In general, the following two LPs are called primal- and dual- LPs to each other.

$$\min \sum_{1 \le j \le n} c_j \cdot x_j \qquad \text{(P)}$$

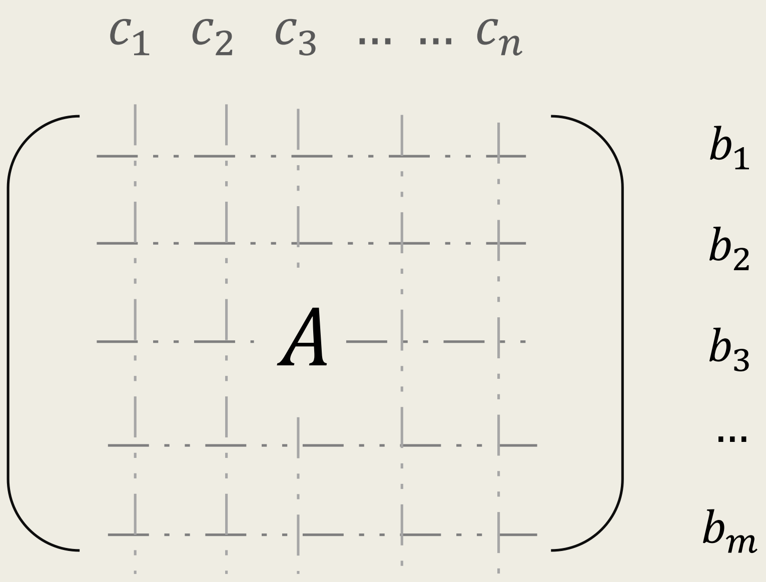
$$\text{s.t.} \sum_{1 \le j \le n} a_{i,j} \cdot x_j \ \ge\ b_i, \qquad \forall 1 \le i \le m,$$

$$x_j \ge 0, \qquad \forall 1 \le j \le n.$$

$$\max \sum_{1 \le i \le m} b_i \cdot y_i \qquad \text{(D)}$$

$$\text{s.t.} \sum_{1 \le i \le m} a_{i,j} \cdot y_i \ \le\ c_j, \qquad \forall 1 \le j \le n,$$

$$y_i \ge 0, \qquad \forall 1 \le i \le m.$$



Primal and Dual LPs

- In general, the following two LPs, (P) and (D), are called primal- and dual- LPs to each other.

$$\min \sum_{1\le j\le n} c_j \cdot x_j \qquad \text{(P)}$$

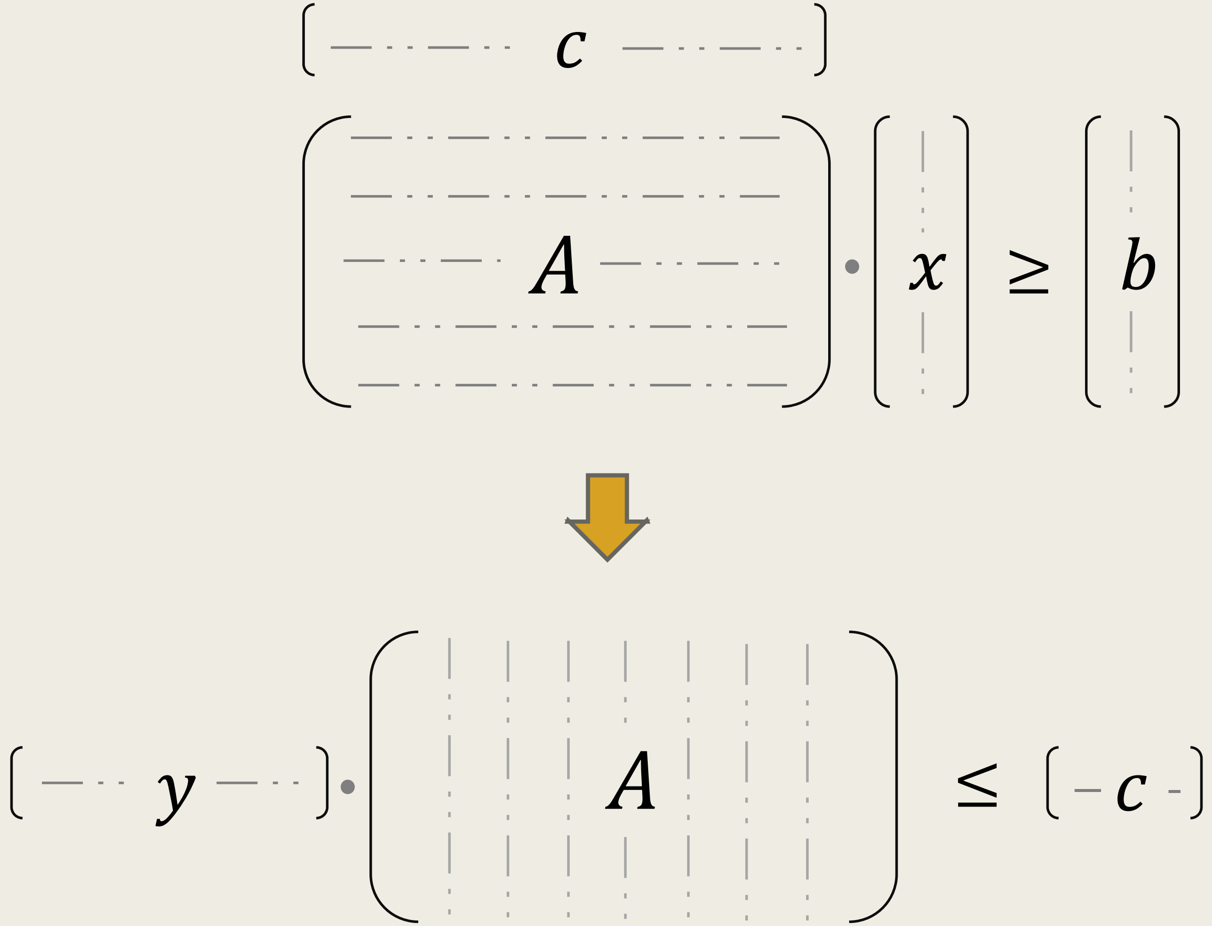
$$\text{s.t.} \sum_{1\le j\le n} a_{i,j} \cdot x_j \ \ge\ b_i, \qquad \forall 1 \le i \le m,$$

$$x_j \ge 0, \qquad \forall 1 \le j \le n.$$

$$\max \sum_{1\le i\le m} b_i \cdot y_i \qquad \text{(D)}$$

$$\text{s.t.} \sum_{1\le i\le m} a_{i,j} \cdot y_i \ \le\ c_j, \qquad \forall 1 \le j \le n,$$

$$y_i \ge 0, \qquad \forall 1 \le i \le m.$$

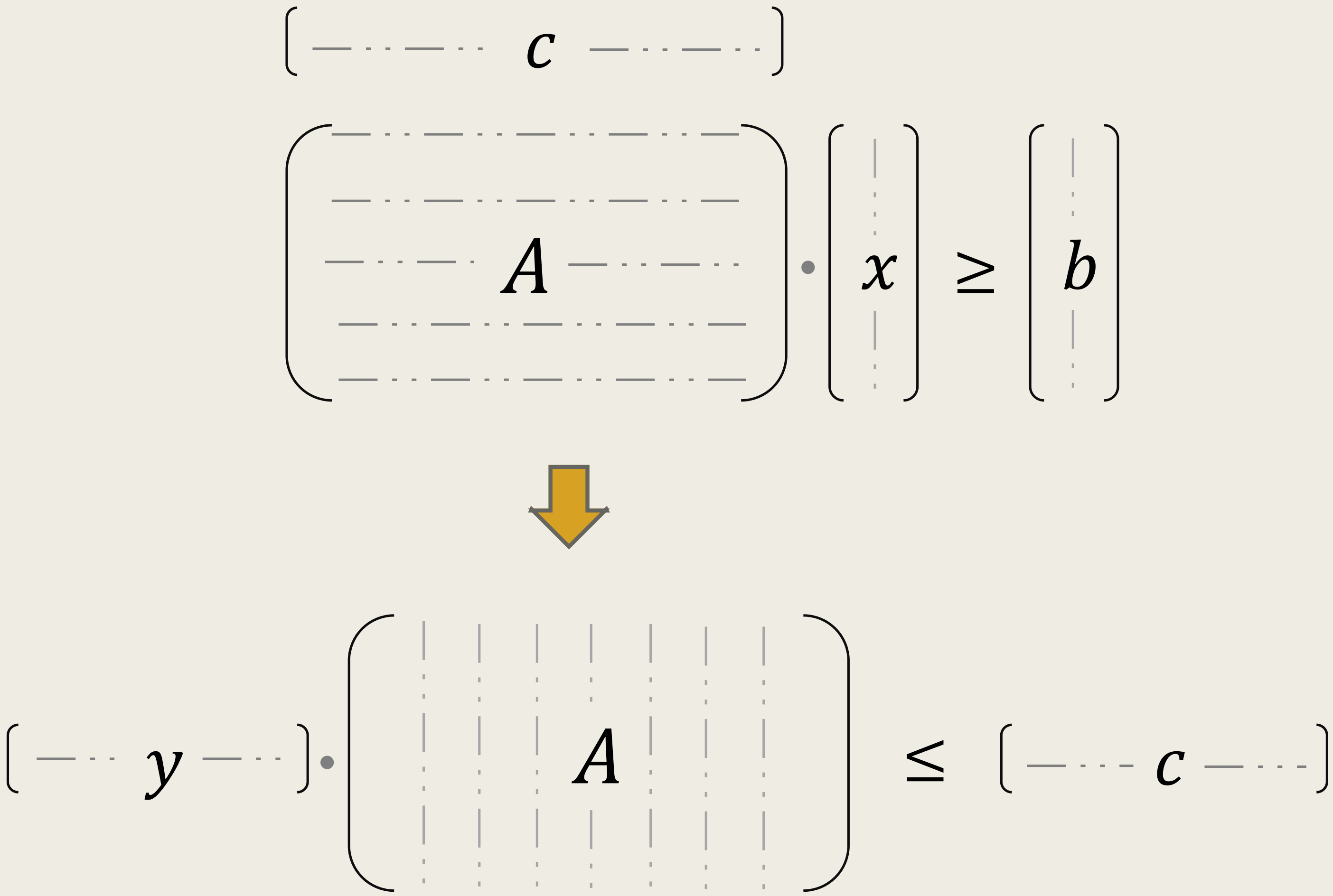


Primal and Dual LPs

- In short, we also write

$$\begin{aligned} \min \quad & \boldsymbol{c} \cdot \boldsymbol{x} & \text{(P)} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b}, & \\ & \boldsymbol{x} \geq 0. & \end{aligned}$$

$$\begin{aligned} \max \quad & \boldsymbol{y} \cdot \boldsymbol{b} & \text{(D)} \\ \text{s.t.} \quad & \boldsymbol{y}A \leq \boldsymbol{c}, & \\ & \boldsymbol{y} \geq 0. & \end{aligned}$$



Primal and Dual LPs

- In general, the following two LPs, (P) and (D), are called primal- and dual- LPs to each other.

$$\begin{aligned} \min \quad & \sum_{1\le j\le n} c_j \cdot x_j && \text{(P)} \\ \text{s.t.} \quad & \sum_{1\le j\le n} a_{i,j} \cdot x_j \ \ge\ b_i, && \forall 1 \le i \le m, \\ & x_j \ge 0, && \forall 1 \le j \le n. \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{1\le i\le m} b_i \cdot y_i && \text{(D)} \\ \text{s.t.} \quad & \sum_{1\le i\le m} a_{i,j} \cdot y_i \ \le\ c_j, && \forall 1 \le j \le n, \\ & y_i \ge 0, && \forall 1 \le i \le m. \end{aligned}$$

- In short, we also write

$$\begin{aligned} \min \quad & \boldsymbol{c} \cdot \boldsymbol{x} && \text{(P)} \\ \text{s.t.} \quad & A\boldsymbol{x} \ \ge\ \boldsymbol{b}, \\ & \boldsymbol{x} \ge 0. \end{aligned}$$

$$\begin{aligned} \max \quad & \boldsymbol{y} \cdot \boldsymbol{b} && \text{(D)} \\ \text{s.t.} \quad & \boldsymbol{y}A \ \le\ \boldsymbol{c}, \\ & \boldsymbol{y} \ge 0. \end{aligned}$$

Weak Duality of LPs

The primal and the dual LPs are in fact bounding each other.

The Weak Duality Theorem

$$\begin{aligned} \min \quad & \boldsymbol{c} \cdot \boldsymbol{x} \qquad \text{(P)} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & \boldsymbol{y} \cdot \boldsymbol{b} \qquad \text{(D)} \\ \text{s.t.} \quad & \boldsymbol{y}A \leq \boldsymbol{c}, \\ & \boldsymbol{y} \geq 0. \end{aligned}$$

- We have seen that, ***any feasible solution of (D)*** corresponds to *a valid way of combining the inequalities of (P)*, and hence gives a ***lower-bound for the optimal value of (P),*** and *vice versa*.
  - This is the weak duality theorem.

The Weak Duality Theorem

$$\begin{aligned} \min \quad & \boldsymbol{c} \cdot \boldsymbol{x} & \text{(P)} \\ \text{s.t.} \quad & A\boldsymbol{x} \geq \boldsymbol{b}, & \\ & \boldsymbol{x} \geq 0. & \end{aligned}$$

$$\begin{aligned} \max \quad & \boldsymbol{y} \cdot \boldsymbol{b} & \text{(D)} \\ \text{s.t.} \quad & \boldsymbol{y}A \leq \boldsymbol{c}, & \\ & \boldsymbol{y} \geq 0. & \end{aligned}$$

Theorem 1.

Let $\boldsymbol{x_0}$ and $\boldsymbol{y_0}$ be feasible solutions for LP-(P) and LP-(D), respectively. Then, we have

$$\boldsymbol{c} \cdot \boldsymbol{x_0} \geq \boldsymbol{y_0} \cdot \boldsymbol{b}\,.$$

By the feasibility of $\boldsymbol{x_0}$ and $\boldsymbol{y_0}$ for LP-(P) and LP(D), we have

$$\boldsymbol{y_0} \cdot \boldsymbol{b} \leq \boldsymbol{y_0} \cdot (A\boldsymbol{x_0}) = (\boldsymbol{y_0}A) \cdot \boldsymbol{x_0} \leq \boldsymbol{c} \cdot \boldsymbol{x_0}.$$

Deriving the Dual LP

Deriving the Dual LP

$$\min \sum_{1 \le j \le n} c_j \cdot x_j \qquad \text{(P)}$$

$$\text{s.t.} \sum_{1 \le j \le n} a_{i,j} \cdot x_j \ge b_i, \qquad \forall 1 \le i \le m,$$

$$x_j \ge 0, \qquad \forall 1 \le j \le n.$$

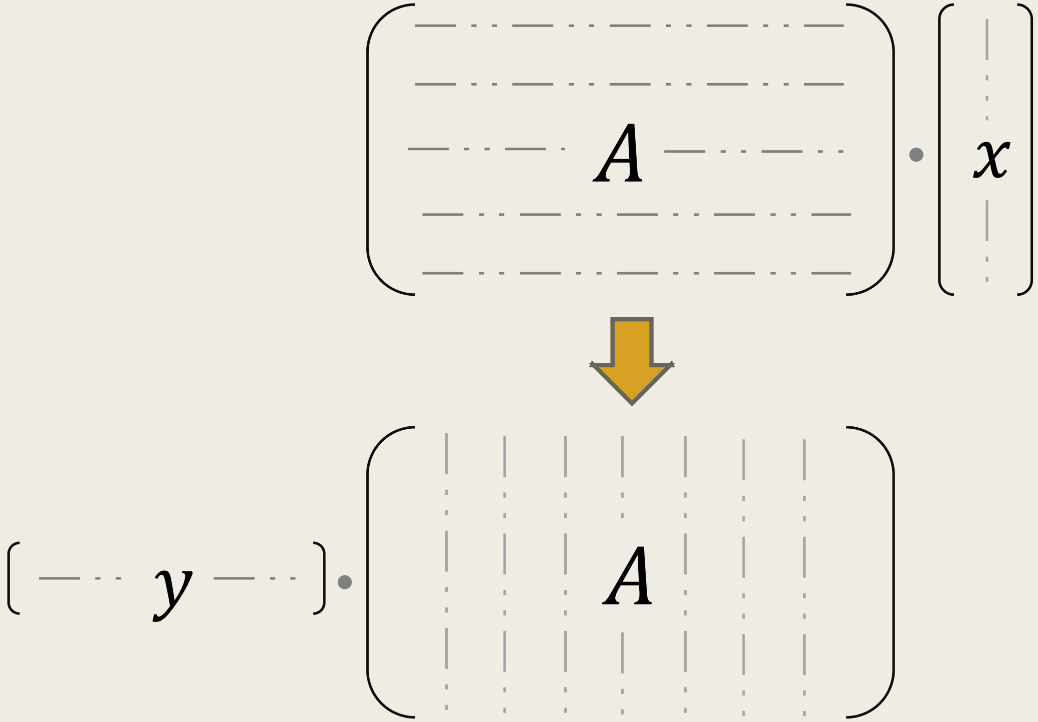
Each variable in (P) corresponds to a ***constraint in (D)***, i.e., a constraint bounding the combined coefficient.

Each constraint in (P) corresponds to ***a variable in (D)***, i.e., the multiplier of the constraints.

$$\max \sum_{1 \le i \le m} b_i \cdot y_i \qquad \text{(D)}$$

$$\text{s.t.} \sum_{1 \le i \le m} a_{i,j} \cdot y_i \le c_j, \qquad \forall 1 \le j \le n,$$

$$y_i \ge 0, \qquad \forall 1 \le i \le m.$$



The Natural LP for Vertex Cover

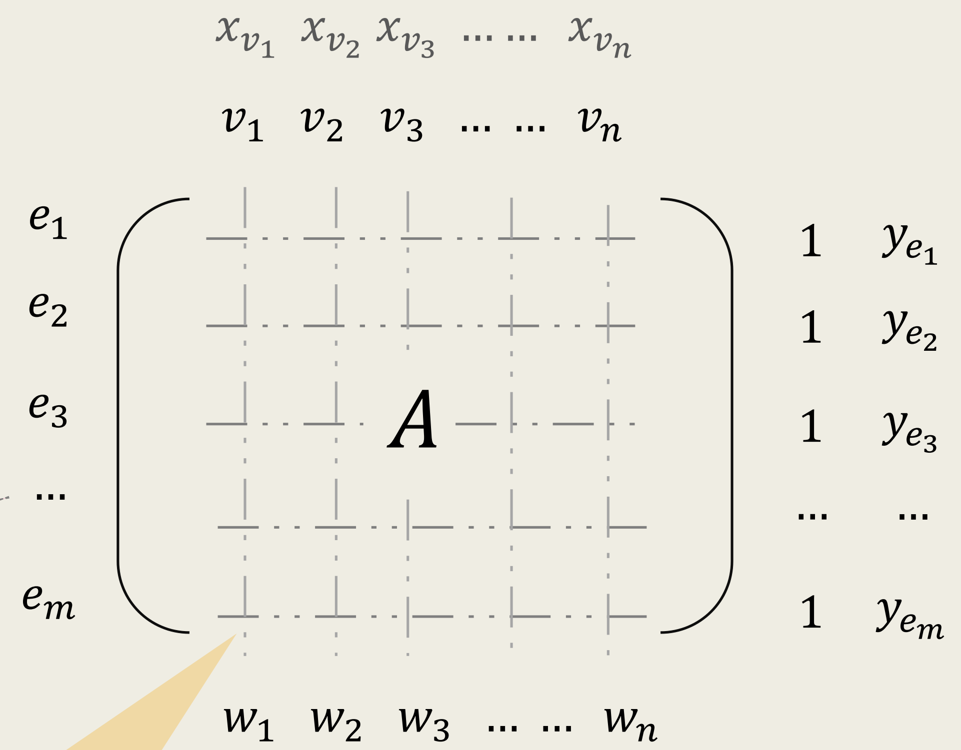
$\forall v \in V$,
a constraint for v

$\forall e = (u, v) \in E$,
a variable y_e

$$\min \sum_{v \in V} w_v \cdot x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1, \qquad \forall (u, v) \in E,$$

$$x_v \geq 0, \qquad \forall v \in V.$$



For each row $e = (u, v) \in E$ in the matrix A,
only column u and v are 1, and the remainings are 0.

0 0 … 0 **1** 0 … 0 **1** 0 … 0 $e = (u, v)$
u v

$$A(e, v) = \begin{cases} 1, & \text{if } v \in e, \\ 0, & \text{otherwise.} \end{cases}$$

$\forall v \in V$, we get a constraint

$$\sum_{e \in E : v \in e} y_e \leq w_v \,.$$

The Dual Natural LP of Vertex Cover

- In conclusion, we get

$$\min \quad \sum_{v \in V} w_v \cdot x_v \qquad \text{(P)}$$

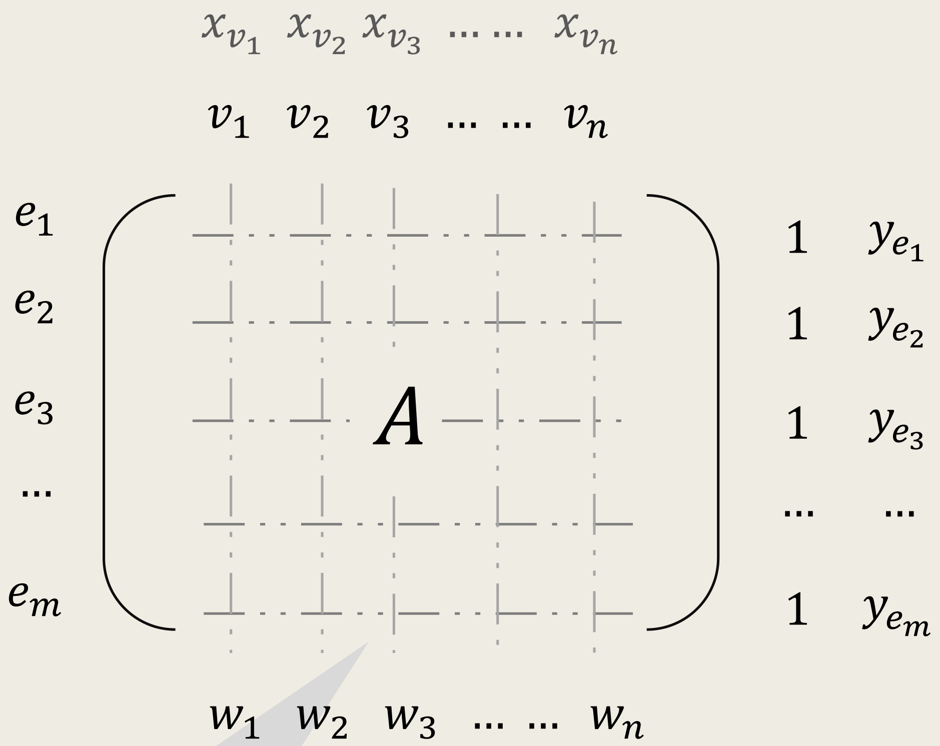
$$\text{s.t.} \quad x_u + x_v \geq 1, \qquad \forall (u,v) \in E,$$

$$x_v \geq 0, \qquad \forall v \in V.$$

$$\max \quad \sum_{e \in E} y_e \qquad \text{(D)}$$

$$\text{s.t.} \quad \sum_{e \in E : v \in e} y_e \leq w_v, \qquad \forall v \in V,$$

$$y_e \geq 0, \qquad \forall e \in E.$$



$$\forall A(e,v) = \begin{cases} 1, & \text{if } v \in e, \\ 0, & \text{otherwise.} \end{cases}$$

$\forall v \in V,$ we get a constraint

$$\sum_{e \in E : v \in e} y_e \leq w_v .$$

Examples of

Natural Primal-Dual Problems

Minimum Vertex Cover & Maximum Matching

- Given a graph $G = (V, E)$, the (cardinality) vertex cover problem is to compute a minimum vertex subset $U \subseteq V$ such that, for any edge $(u, v) \in E$, u or v is in U.

$$\begin{aligned} \min \quad & \sum_{v \in V} x_v & \text{(P)} \\ \text{s.t.} \quad & x_u + x_v \geq 1, \quad \forall (u, v) \in E, \\ & x_v \geq 0, \quad \forall v \in V. \end{aligned}$$

⟺

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e & \text{(D)} \\ \text{s.t.} \quad & \sum_{e \in E : v \in e} y_e \leq 1, \quad \forall v \in V, \\ & y_e \geq 0, \quad \forall e \in E. \end{aligned}$$

- The (cardinality) maximum matching problem is to compute a maximum size edge subset $M \subseteq E$ such that each vertex $u \in V$ is incident to at most edge in M.

Minimum Vertex Cover & Maximum Matching

- Given a graph $G = (V, E)$, the (***weighted***) vertex cover problem is to compute a minimum vertex subset $U \subseteq V$ such that, for any edge $(u, v) \in E$, u or v is in U.

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v \cdot x_v & \text{(P)} \\ \text{s.t.} \quad & x_u + x_v \geq 1, \quad \forall (u, v) \in E, \\ & x_v \geq 0, \quad \forall v \in V. \end{aligned}$$

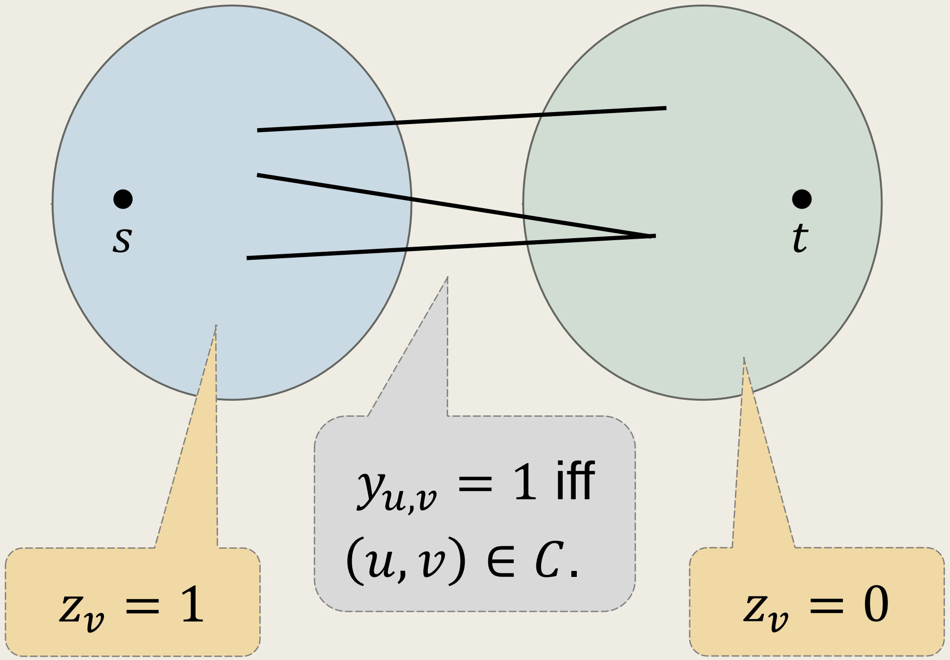
$\Leftrightarrow$

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e & \text{(D)} \\ \text{s.t.} \quad & \sum_{e \in E : v \in e} y_e \leq w_v, \quad \forall v \in V, \\ & y_e \geq 0, \quad \forall e \in E. \end{aligned}$$

- The (**weighted**) maximum matching problem is to compute a maximum size edge *multi-subset* $M \subseteq E$ such that each vertex $u \in V$ is incident to *at most* w_v *edges in* M.

Max-Flow & Min-Cut

- The natural LPs for maximum flow and minimum cut.



$$\max \sum_{v \in V:(s,v)\in E} f_{s,v} \qquad \text{(F)}$$

$$\text{s.t.} \quad \sum_{u \in V:(u,v)\in E} f_{u,v} - \sum_{w \in V:(v,w)\in E} f_{v,w} = 0, \qquad \forall v \in V - \{s,t\},$$

$$0 \le f_{u,v} \le c_{u,v}, \qquad \forall (u,v) \in E.$$

$$\min \sum_{(u,v)\in E} c_{u,v} \cdot y_{u,v} \qquad \text{(C)}$$

$$\begin{aligned}
\text{s.t.} \quad & y_{u,v} + z_v \ge 1, && \forall v \in V: (s,v) \in E, \\
& y_{u,v} - z_v \ge 0, && \forall v \in V: (v,t) \in E, \\
& y_{u,v} - z_u + z_v \ge 0, && \forall u,v \in V - \{s,t\}: (u,v) \in E, \\
& y_{u,v} \ge 0, && \forall (u,v) \in E. \\
& z_v \in \mathbb{R}, && \forall v \in V - \{s,t\}.
\end{aligned}$$

2-Approximation for Vertex Cover

via the Dual-Fitting Technique

The Dual-Fitting Technique

- Consider the primal and dual LPs for vertex cover.

$$\min \sum_{v \in V} w_v \cdot x_v \qquad \text{(P)}$$
$$\text{s.t.} \quad x_u + x_v \geq 1, \qquad \forall (u,v) \in E,$$
$$x_v \geq 0, \qquad \forall v \in V.$$

$$\max \sum_{e \in E} y_e \qquad \text{(D)}$$
$$\text{s.t.} \quad \sum_{e \in E : v \in e} y_e \leq w_v, \qquad \forall v \in V,$$
$$y_e \geq 0, \qquad \forall e \in E.$$

  - By the weak duality theorem,
    any feasible solution for LP-(D) gives a lower-bound on LP-(P).
  - In this part, we present a simple process that computes a feasible solution for LP-(D) that also corresponds to a 2-approximation for VC.

The Dual-Fitting Technique

- The idea is to compute a (maximal) feasible solution of LP-(D).
  - We start with a trivial solution $\boldsymbol{y} = 0$ and gradually increase its value.
  - When a vertex inequality becomes tight,
    the cost of that vertex ***<u>can be paid</u>*** by the dual values of its incident edges.
- During the process, a feasible integral solution for LP-(P) is also formed.

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v \cdot x_v && \text{(P)} \\ \text{s.t.} \quad & x_u + x_v \geq 1, \quad \forall (u,v) \in E, \\ & x_v \geq 0, \quad \forall v \in V. \end{aligned}$$

$$\begin{aligned} \max \quad & \sum_{e \in E} y_e && \text{(D)} \\ \text{s.t.} \quad & \sum_{e \in E : v \in e} y_e \leq w_v, \quad \forall v \in V, \\ & y_e \geq 0, \quad \forall e \in E. \end{aligned}$$

The Dual-Fitting Process for LP-(D)

- The simple process goes as follows.
  - $\hat{\boldsymbol{y}} \leftarrow \mathbf{0}$,
    $E' \leftarrow E, \quad V' \leftarrow V$.
  - While $E' \neq \emptyset$, do
    - **Raise the variable $\hat{y}_e$ for all $e \in E'$ simultaneously at the same rate** until the inequality $\sum_{e \in E : v \in e} \hat{y}_e \le w_v$ for some $v \in V'$ holds with equality.
      Let $U \subseteq V'$ denote the set of vertices whose inequalities are tight and $E[U]$ denote the set of incident edges of U.
    - $V' \leftarrow V' - U$.
      $E' \leftarrow E' - E[U]$.
  - Output $\mathcal{C} \coloneqq V - V'$.

$$\max \quad \sum_{e \in E} y_e \qquad \text{(D)}$$

$$\text{s.t.} \quad \sum_{e \in E : v \in e} y_e \le w_v, \qquad \forall v \in V,$$

$$y_e \ge 0, \qquad \forall e \in E.$$

This process ***greedily*** *pack the values into the dual variables* until the constraints are tight.

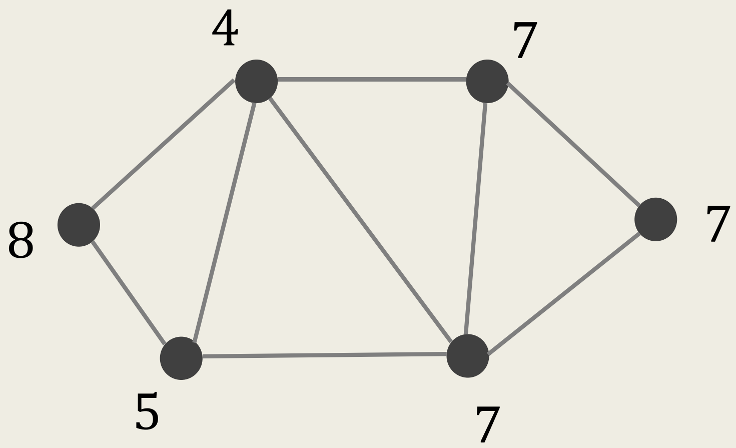
$$\max \sum_{e \in E} y_e \qquad \text{(D)}$$

$$\text{s.t.} \sum_{e \in E : v \in e} y_e \le w_v, \qquad \forall v \in V,$$

$$y_e \ge 0, \qquad \forall e \in E.$$

Example

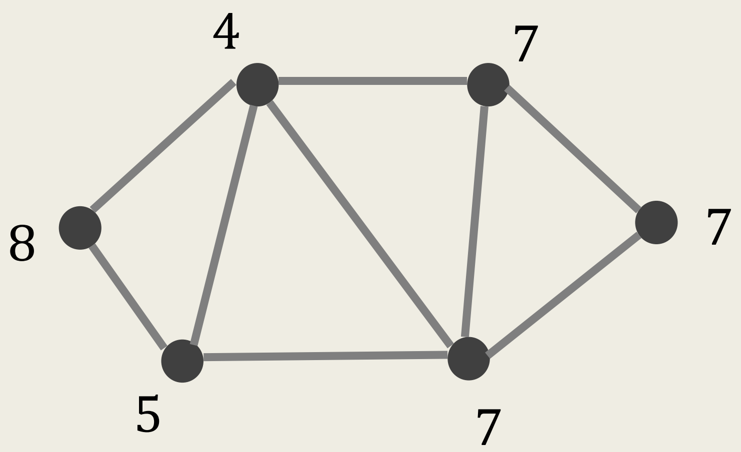
- Consider the following example.



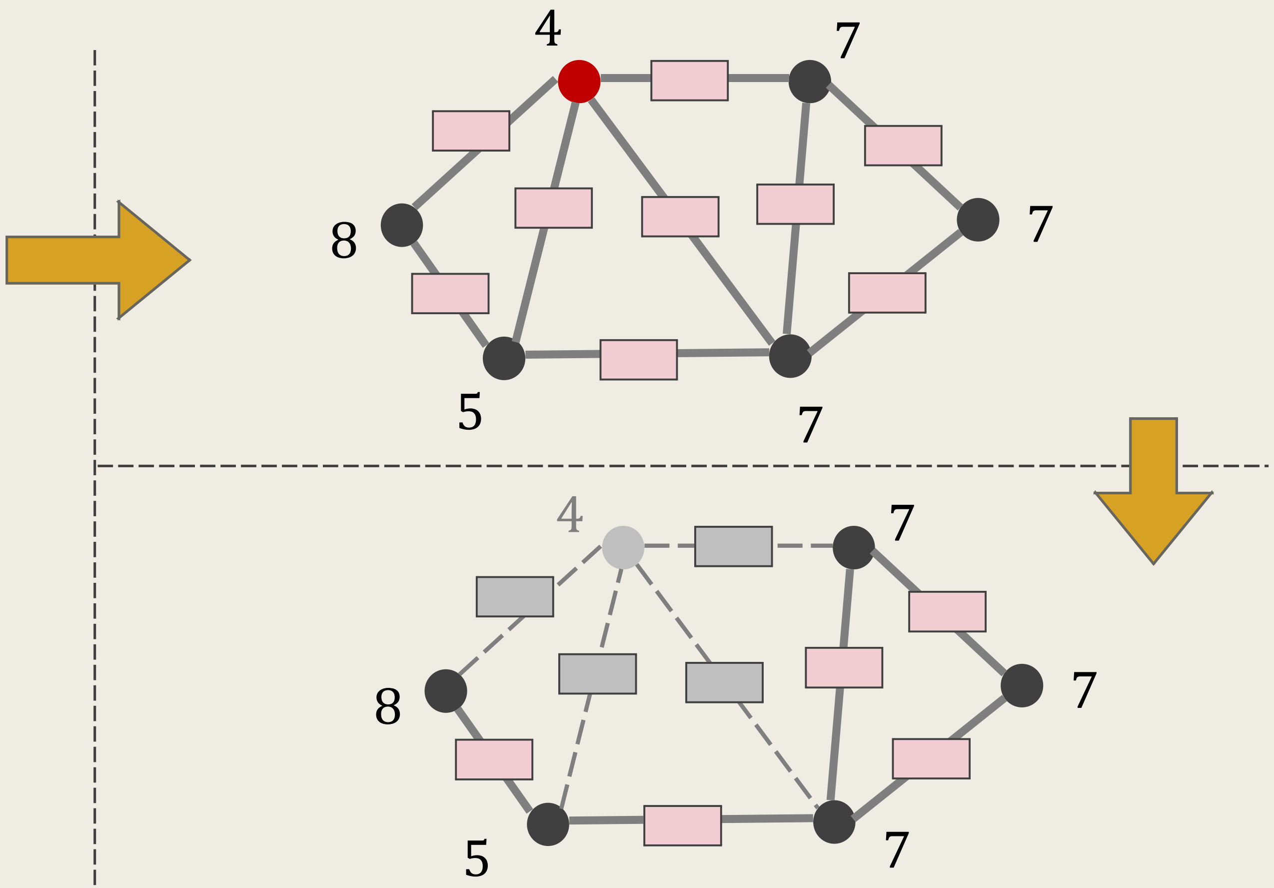
Example

- Consider the following example.

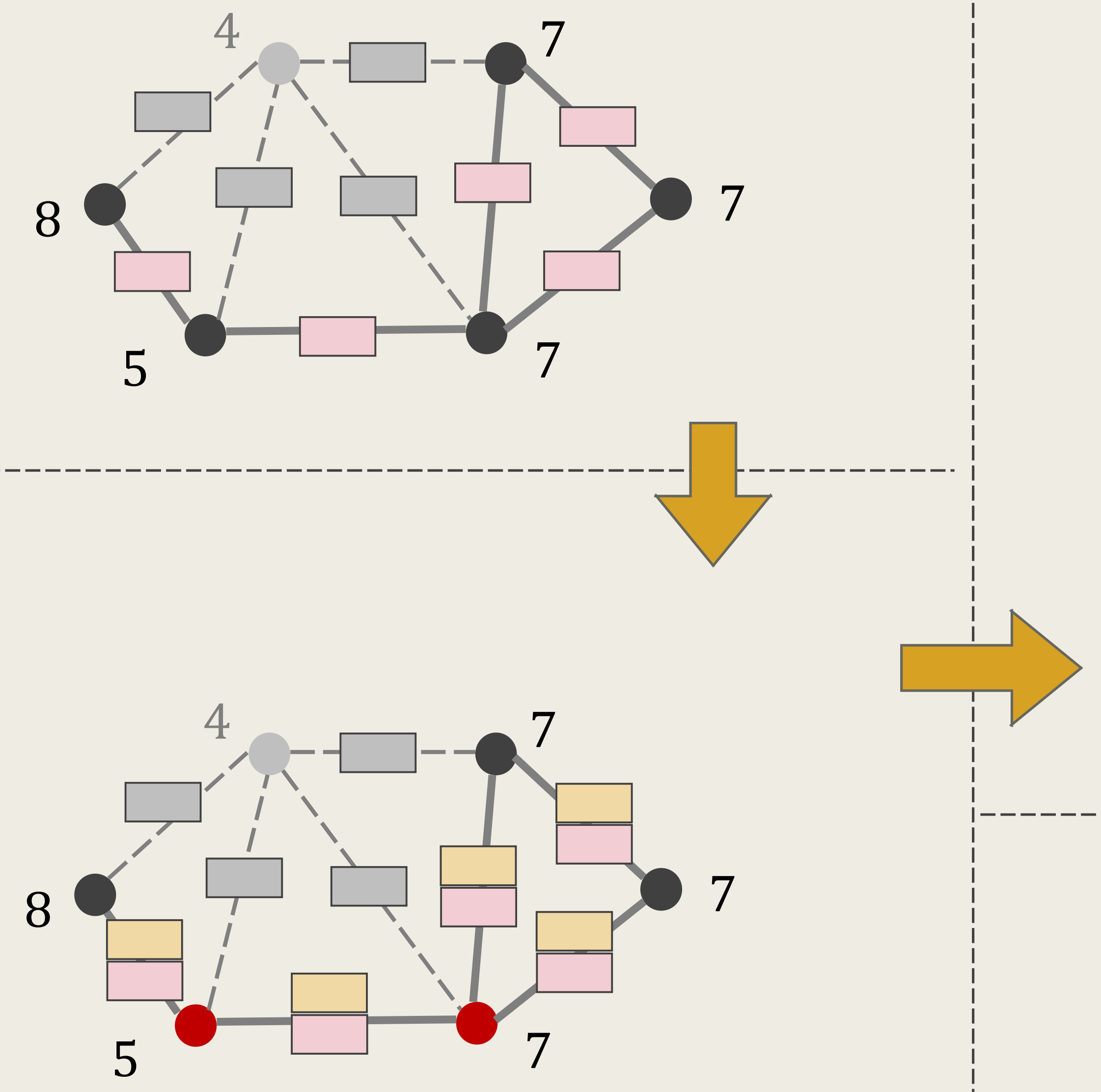
$$\max \quad \sum_{e \in E} y_e \qquad \text{(D)}$$
$$\text{s.t.} \quad \sum_{e \in E : v \in e} y_e \le w_v, \qquad \forall v \in V,$$
$$y_e \ge 0, \qquad \forall e \in E.$$

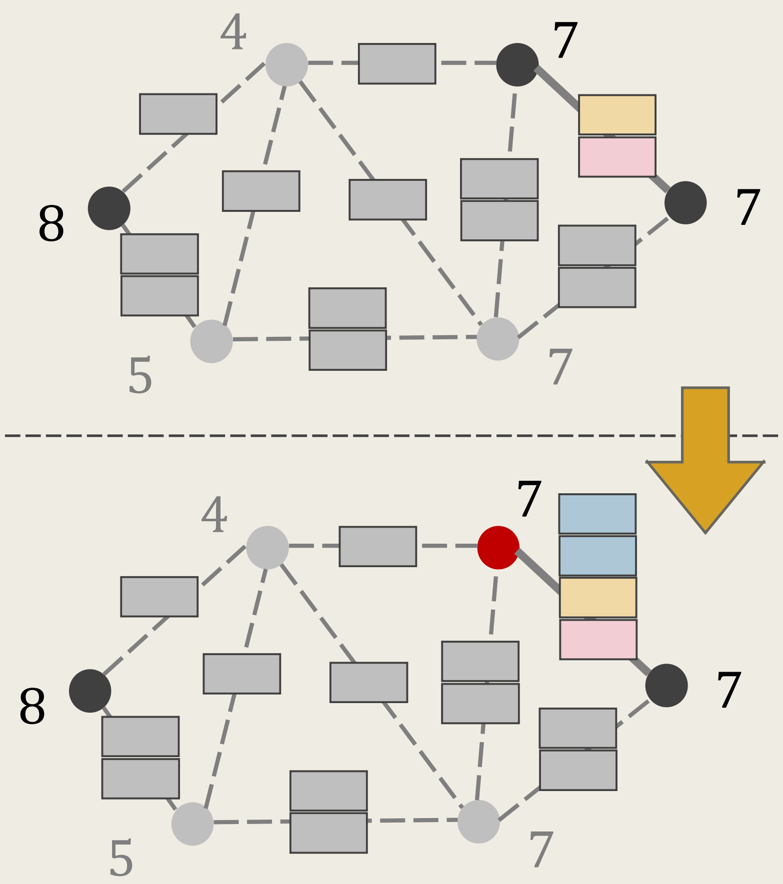


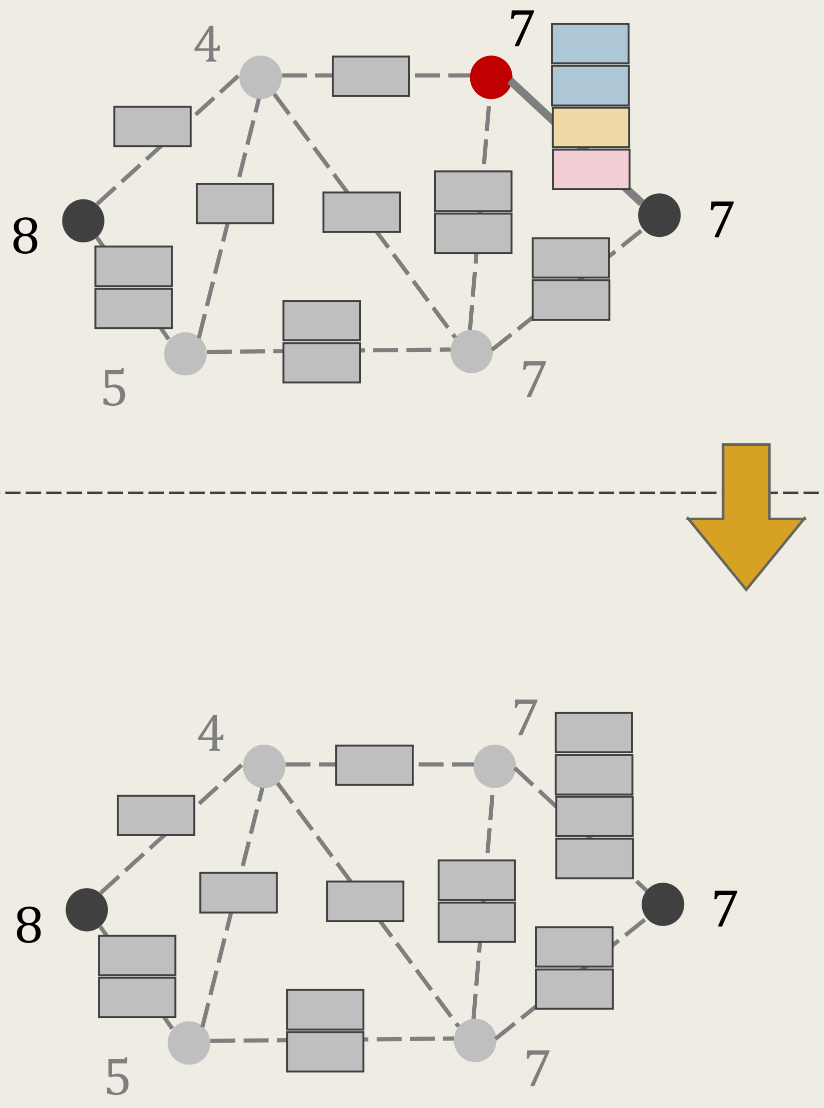
$$E' := E, \qquad V' := V.$$



$$
\begin{aligned}
\max \quad & \sum_{e \in E} y_e && \text{(D)} \\
\text{s.t.} \quad & \sum_{e \in E : v \in e} y_e \le w_v, \qquad \forall v \in V, \\
& y_e \ge 0, \qquad \forall e \in E.
\end{aligned}
$$







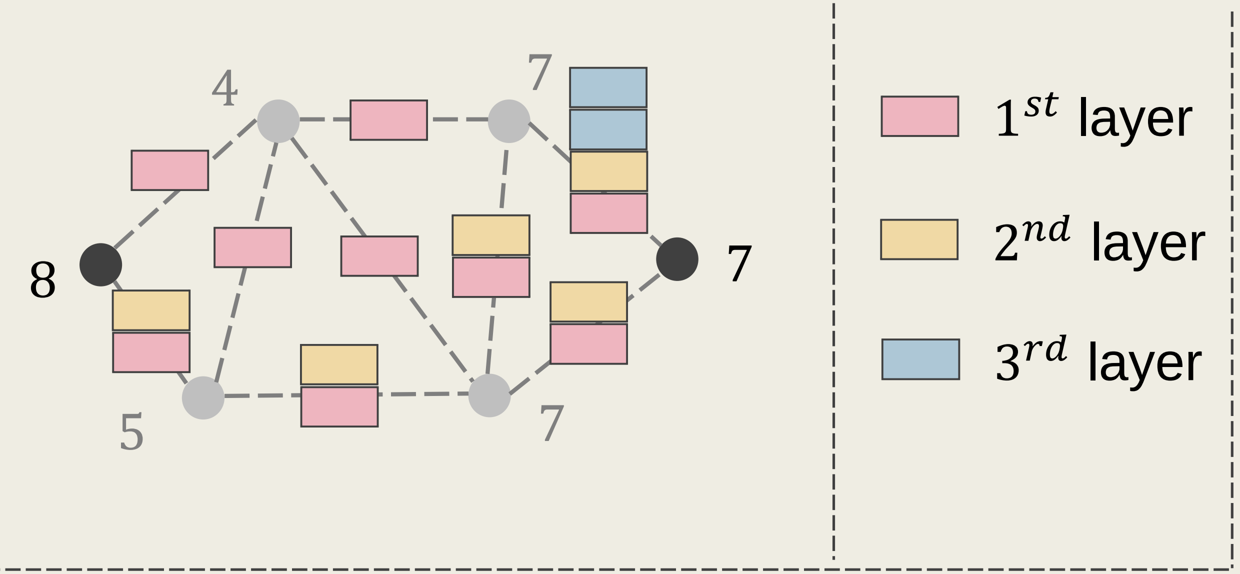
$$\max \quad \sum_{e \in E} y_e \qquad \text{(D)}$$

$$\text{s.t.} \quad \sum_{e \in E : v \in e} y_e \le w_v, \qquad \forall v \in V,$$

$$y_e \ge 0, \qquad \forall e \in E.$$

- For each v selected,
  we have $\sum_{e \in E : v \in e} y_e = w_v$.
- Each edge pays for at most two vertices.
- So, the total cost of the selected vertices is at most

$$2 \cdot \sum_{e \in E} y_e \le 2 \cdot OPT_f \,.$$



$$\begin{aligned} \max \quad & \sum_{e \in E} y_e & & \text{(D)} \\ \text{s.t.} \quad & \sum_{e \in E : v \in e} y_e \le w_v, & \forall v \in V, & \\ & y_e \ge 0, & \forall e \in E. & \end{aligned}$$

- We can also observe that,
  the process of raising the dual variables is equivalent to
  defining the *"degree-weighted functions"* in the layering algorithm.

- The layering algorithm is in fact a dual-fitting algorithm.
  - Its behavior become much simpler when we look at it from the perspective of LP duality.

The Analysis – Feasibility

- During the process,
  the following invariant holds in the beginning of each while loop.
  - For any $e = (u, v) \in E'$, we have $u, v \in V'$.

- Consider each while loop.
  - When the value of $\hat{y}_e$ is raised for each $e \in E'$,
    the inequality of each $v \in V'$ is becoming tighter and
    some inequality will hold with equality.
  - So, at least one vertex along with its incident edges will be removed.
  - The invariant holds after each loop.

- Hence, when it ends, E' is empty and we have a feasible vertex cover.

The Analysis – Approximation Guarantee

- For the guarantee of the output,
  observe that $\hat{y}$ is a feasible solution for LP-(D).
  - By the weak duality, we have $\sum_{e\in E}\hat{y}_e \le OPT\big(\mathrm{LP{-}(P)}\big) \le OPT$.
- We have

$$w(\mathcal{C}) = \sum_{v\in V-V'} w(v) = \sum_{v\in V-V'}\ \sum_{e\in E:v\in e} \hat{y}_e \le 2\cdot\sum_{e\in E}\hat{y}_e \le 2\cdot OPT\,.$$

By the dual-fitting process, each $v\in V-V'$ has its inequality hold with equality.

Each $\mathrm{e}\in E$ is counted **at most twice** in the summation.

By the weak duality.

Implementing the Dual-Fitting Process to Run in Polynomial-Time

$$
\begin{aligned}
\max \quad & \sum_{e \in E} y_e && && \text{(D)} \\
\text{s.t.} \quad & \sum_{e \in E : v \in e} y_e \le w_v, && \forall v \in V, \\
& y_e \ge 0, && \forall e \in E.
\end{aligned}
$$

- The simple process goes as follows.
  - $\boldsymbol{w}' \leftarrow \boldsymbol{w}$,
    $E' \leftarrow E, \quad V' \leftarrow V$.
  - While $E' \neq \emptyset$, do
    - Let $t \leftarrow \min_{v \in V'} w'(v) / \deg_{E'}(v)$.
    - For each $v \in V'$, set $w'(v) \leftarrow w'(v) - t \cdot \deg_{E'}(v)$.
      Let $U \coloneqq \{\, v \in V' : \ w'(v) = 0 \,\}$.
    - $V' \leftarrow V' - U$.
      $E' \leftarrow E' - E[U]$.
  - Output $\mathcal{C} \coloneqq V - V'$.

This is exactly ***the layering algorithm***, interpreted in the language of ***LP dual-fitting***.