The LP relaxation.

(*).

$$\min \sum_{e \in E} c_e \cdot x_e$$

$$\text{s.t.} \sum_{e \in \delta(S)} x_e \geq f(S). \quad \forall S \subseteq V.$$

$$0 \leq x_e \leq u_e. \quad \forall e \in E.$$

cut requirement function

$\forall S \subseteq V.$

$$f(S) \equiv \max_{\substack{u \in S \\ v \notin S}} r(u.v)$$

weakly supermodular.

$\Updownarrow$

① $f(V) = 0$.

② $\forall A. B \subseteq V$. at least one of the followings holds.

(i) $f(A) + f(B) \leq f(A-B) + f(B-A)$.

(ii) $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$.

Theorem. 1.

$\forall$ weakly supermodular function,

any extreme point solution x for (*).

with $0 < x_e < 1$. $\forall e \in E$.

there exists $e \in E$. s.t. $x_e \geq \frac{1}{2}$.

*. Definition.

A set family $\mathcal{L} \subseteq 2^V$ is laminar.

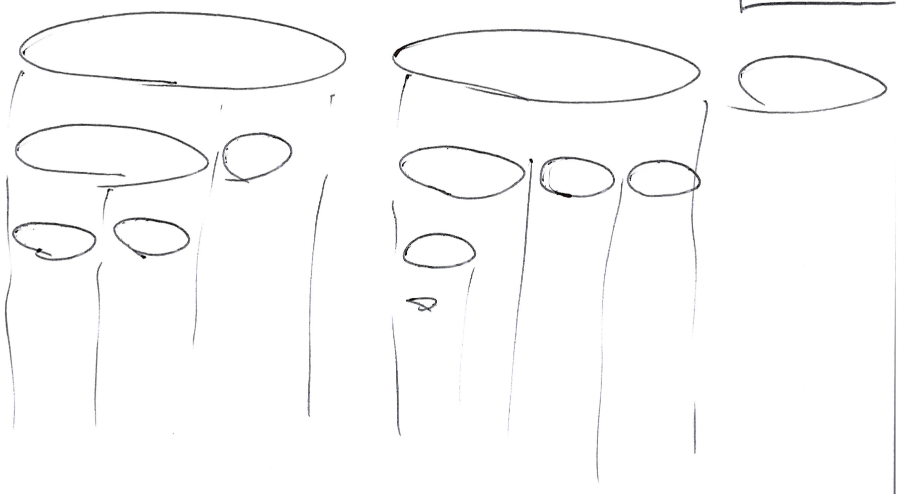
if for every $A, B \in \mathcal{L}$ with $A \neq B$.

exactly one of the followings holds.

① $A \cap B = \phi$.

② $A \subseteq B$.

③ $B \subseteq A$.



*. Theorem. 2.

For any extreme point solution x for (*).

there exists a collection $\mathcal{L}$ of m tight sets s.t. ① $\{ v_S : S \in \mathcal{L} \}$ is independent.

② $\mathcal{L}$ is laminar. automatic

For any $S \subseteq V$.

define $v_S \in \mathbb{R}^m$ be the incidence vector of $\delta(S)$.

$$\boxed{\sum_{e \in \delta(S)} x_e = \underbrace{v_S \cdot x}_{\equiv\ \delta_x(S)}}.$$

*. To prove Theorem 2.

① Start with $\mathcal{L}=\phi$.

② While $|\mathcal{L}|<m$.

(a). Pick a tight set $S \notin \mathrm{Span}(\mathcal{L})$.

(b). While $cr(S,\mathcal{L})>0$.

- Pick tight S' s.t. $S' \notin \mathrm{Span}(\mathcal{L})$, $cr(S',L) < cr(S,L)$.
- Set $S \leftarrow S'$.

Guaranteed by the following lemmas.

(c) Add S to $\mathcal{L}$.

③ Output $\mathcal{L}$.

$\forall S \subseteq V$.

$cr(S,L) = |\{A \in L : A \text{ crosses } S\}|$.

$\forall A,B \subseteq V$.

A crosses B if ① $A \cap B \neq \phi$. ② $A-B \neq \phi$. $B-A \neq \phi$.

Lemma. 3. Let A. B be two crossing tight sets.

Then one of the followings holds.

Crossing $\Rightarrow$ non-crossing.

remains linearly independent.

① $A-B$ and $B-A$ are tight. $\chi_A + \chi_B = \chi_{A-B} + \chi_{B-A}$.

② $A\cap B$ and $A\cup B$ are tight. $\chi_A + \chi_B = \chi_{A\cup B} + \chi_{A\cap B}$.

Lemma 4. Let S be a set that crosses some $T \in \mathcal{L}$.

Then.

$$\max \left\{ \begin{array}{l} cr(S-T, \mathcal{L}) \\ cr(T-S, \mathcal{L}) \\ cr(S\cup T, \mathcal{L}) \\ cr(S\cap T, \mathcal{L}) \end{array} \right\} < cr(S, \mathcal{L}).$$

$(S, T) \Rightarrow (S-T, \boxed{T-S})$

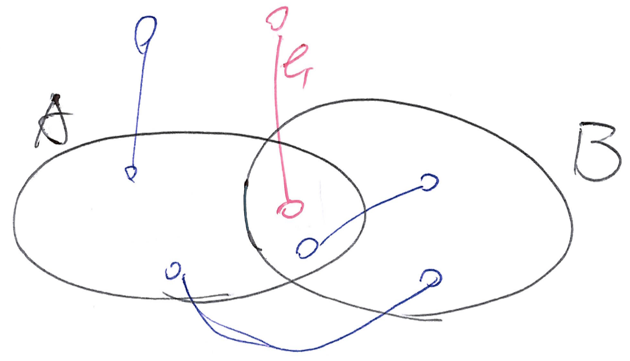
or $(S\cup T, \boxed{S\cap T})$.

Still laminar.

Proof of lemma 3.

4-9 | 10

Since f is weakly supermodular.



either $f(A)+f(B) \le f(A-B)+f(B-A)$.

or $f(A)+f(B) \le f(A\cup B)+f(A\cap B)$.

① Suppose that $\underbrace{f(A)+f(B)}_{\shortparallel} \le \underbrace{f(A-B)}_{\le \delta_X(A-B)} + \underbrace{f(B-A)}_{\le \delta_X(B-A)}$.

then.

$\delta_X(A)+\delta_X(B)$

$$\Rightarrow \delta_X(A)+\delta_X(B) \le \delta_X(A-B)+\delta_X(B-A).$$

e_1 type contribute more to the left.

the remainings contribute the same to both.

② Symmetric.

$$\Rightarrow \delta_X(A)+\delta_X(B) = \delta_X(A-B)+\delta_X(B-A).$$

$X_e>0.\ \forall e.$ $\Rightarrow v_A+v_B = v_{A-B}+v_{B-A}$. #

Proof of lemma 4.

① Consider $S-T$.

(i). $S-T$ does not cross T.

(ii) $\forall T' \in \mathcal{L}$. s.t.

T' crosses $S-T$.

T' crosses S too.

$\Rightarrow \; cr(S-T, \mathcal{L}) < cr(S, \mathcal{L})$.

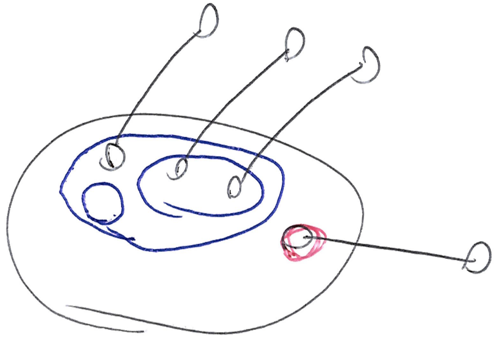
② The remaining similar.

* A counting argument.

- Consider the forest formed by $\mathcal{L}$.

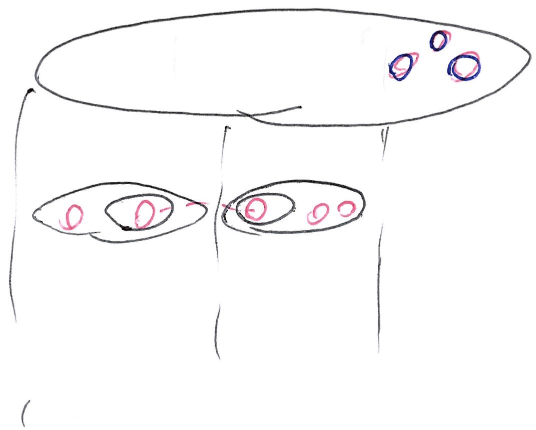
- For any $S \in \mathcal{L}$.

  consider edges in $\delta(S)$.



  - S owns an endpoint v of $e = (u, v) \in \delta(S)$.

    if S is the smallest set in $\mathcal{L}$ containing v.



  - Consider the subtree rooted at S.

    the subtree owns all the vertices owned by its descendants.

- There are m edges with $2m$ endpoints.

- $\forall e \in E$. $0 < x_e < \frac{1}{2}$.

  define $y_e = \frac{1}{2} - x_e \Rightarrow 0 < y_e < \frac{1}{2}$.

  $\forall S \in \mathcal{L}$. define co-requirement

$$\text{coreq}(S) \equiv \sum_{e \in \delta(S)} y_e = \frac{1}{2}|\delta(S)| - \underbrace{f(S)}_{v_S \cdot x = \sum_{e \in \delta(S)} x_e}.$$

- Then. ① $0 < \text{coreq}(S) < \frac{1}{2}|\delta(S)|$.

  ② $\text{coreq}(S) \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\}$ is semi-integral

  iff. $|\delta(S)|$ is odd. Otherwise, $\text{coreq}(S)$ is integral.

* Consider a set $S \in \mathcal{L}$, and the subtree T rooted at S.

<u>Lemma. 5</u>

The vertices owned by T can be redistributed in a way such that ① S gets at least 3 vertices.

② Each of the proper descendants of S gets 2 vertices.

Furthermore, if $coreq(S) \neq \frac{1}{2}$, then S must get at least 4 vertices.

---

* By lemma 5,

the total number of vertices is at least $2|\mathcal{L}|+1 \geq 2m+1$.

a contradiction.

Proof of Lemma 5.

$\mathcal{L}\delta$

① if S is a leaf.

then $|\delta(S)| \geq 3$. since $0 < x_e < \frac{1}{2}$. $\forall e$ and $f(S) \neq 0$ is integral. $\Rightarrow$ S owns at least 3 vertices.

if $|\delta(S)| = 3$.

then $0 < \text{coreq}(S) = \frac{1}{2}|\delta(S)| - \overset{U_S \cdot y}{\sum_{e \in \delta(S)} y_e} < \frac{3}{2}$.

$\Rightarrow \text{coreq}(S) = \frac{1}{2}$.

Have. $\text{coreq}(S) \neq \frac{1}{2}$

$\Rightarrow |\delta(S)| \geq 4$.

S owns at least 4.

② if S has 4 or more children.

$\Rightarrow$ assign one surplus from each of them to S. done.

③. if S has 3 children.

(a) if some of them has a surplus of 2.
or S owns at least 1. $\Rightarrow$ done.

(b) otherwise, $coreq(S') = \frac{1}{2}$ $\forall$ child S' of S.

$\Rightarrow$ By lemma 6. $coreq(S) = \frac{1}{2}$.

$\Rightarrow$ 3 vertices suffice for S.

Lemma 6

If ① S has α children.
owns β vertices.
with $\alpha + \beta = 3$
② $coreq(S') = \frac{1}{2}$
$\forall$ child s' of S.
then $Coreq(S) = \frac{1}{2}$.

④. if S has 2 children.

(a) if each has a surplus of 2. $\Rightarrow$ done.

(b) if each children has a surplus 1.

$\Rightarrow$ $coreq(S) = \frac{1}{2}$. if S only owns 1 vertex.
done.

⑤ if S has only one child,
similar.

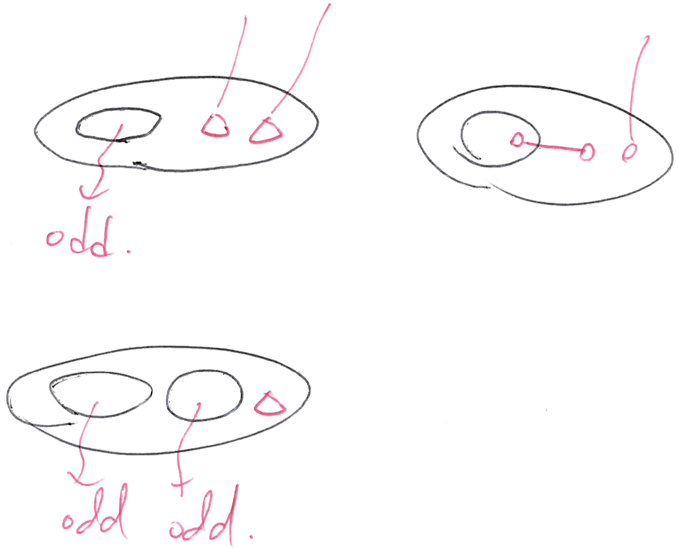
Lemma 7.

If S has 2 children,
one of which has
$Coreq.\ \frac{1}{2}$.
then S owns at least
one vertex.

Lemma 8.

If S has one child.
then it owns at least 2 vertices.

Proof of Lemma 6.



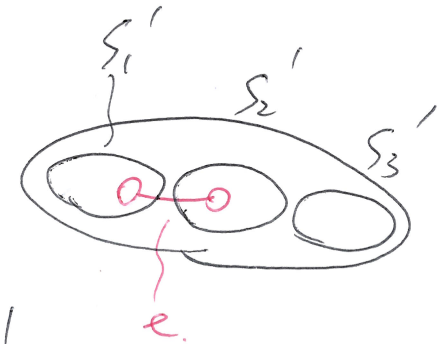
*. Since $\mathrm{coreq}(S') = \frac{1}{2}$ $\forall$ child S' of S.

$|\delta(S')|$ is odd.

$\Rightarrow$ $|\delta(S)|$ odd.

$\Rightarrow$ $\mathrm{coreq}(S)$ semi-integral.

$$\mathrm{coreq}(S) = \sum_{e \in \delta(S)} y_e \le \sum_{S' \text{ child of } S} \mathrm{coreq}(S') + \sum_{\substack{e \\ \text{with one endpoint} \\ \text{owned by } S}} y_e$$



① $\beta > 0$. $\Rightarrow$ $\mathrm{coreq}(S) < \frac{1}{2}(\alpha+\beta) = \frac{3}{2}$. since $y_e < \frac{1}{2}$.

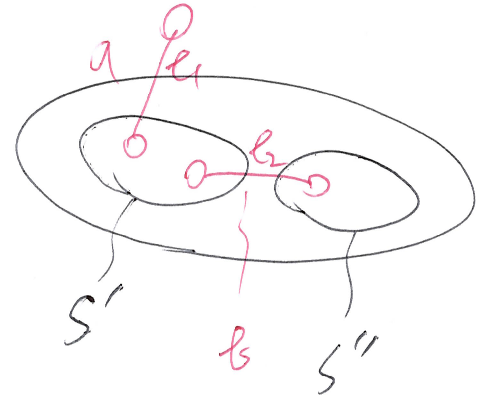
② $\beta = 0$. there must exist edges like e.

since $v_{S_1'}$, $v_{S_2'}$, $v_{S_3'}$, v_S are linearly independent.

$\Rightarrow$ $\mathrm{coreq}(S) < \sum_{S'} \mathrm{coreq}(S') = \frac{3}{2}$.

Proof of Lemma 7.

Let S', S'' be children of S. with $\mathrm{coreq}(S') = \frac{1}{2}$.



Suppose that S owns no vertex.

Since $U_{S'}, U_{S''}, U_S$ are linearly independent, there must exist edges like e_1, e_2.

let $a = \sum_{e \in E_1} y_e > 0$, $b = \sum_{e \in E_2} y_e > 0$.

$\Rightarrow \mathrm{coreq}(S') = a + b = \frac{1}{2}$. $\Rightarrow$ $|a - b| < \frac{1}{2}$.

$a > 0$. $b > 0$.

(i). $\mathrm{coreq}(S) - \mathrm{coreq}(S'') = a - b$.

$= 0$.

$\Rightarrow |\delta(S)|, |\delta(S'')|$ have the same parities.

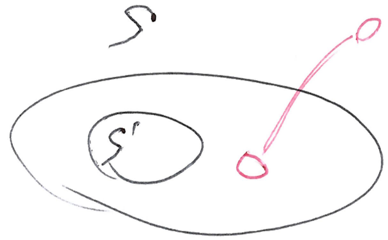
(ii). $\mathrm{coreq}(S') = \frac{1}{2}$.

$\Rightarrow |\delta(S')|$ is odd.

$\Rightarrow |\delta(S)|, |\delta(S'')|$ have different parities $\times$.

Proof of Lemma 8.

Let S' be the child of S.



— S owns at least one vertex.

since $U_S \neq U_{S'}$.

— if S owns only one vertex.

$\Rightarrow \left| \delta_x(S) - d_x(S') \right| < \frac{1}{2}$. $\times$.

since they are tight. $f(S)$. $f(S')$ integral.