*. The Steiner Network Problem.

Given $G=(V,E)$ with.

① $c: E \to Q^+$ edge cost function.

② $r: V \times V \to Z^+$ connectivity requirement.

③ $u: E \to Z^+ \cup \{\infty\}$. multiplicity limit,

Find $x: E \to Z^+ \cup \{0\}$ edge multiplicity function

s.t. $x \le e$. ~~there are r(u.v~~)

G_x satisfies the connectivity constraint r.

$\to$ $\exists$ $r(u.v)$ edge disjoint paths between $u.v$. $\forall u.v$.

and $\sum_{e \in E} c(e) \cdot x_e$. is minimized.

(1

$E_x = \{\underbrace{e, e, \cdots, e}_{x_e} \mid e \in E\}$.

$G_x = (V, E_x)$.

LP relaxation.

(*)

$$\min \sum_{e \in E} c_e \cdot x_e.$$

$$\sum_{e:\, e \in \delta(S)} x_e \geq f(S). \quad \forall S \subseteq V.$$

$$0 \leq x_e \leq u_e. \quad \forall e \in E.$$

Theorem. 1. $\boxed{\forall \text{ weakly supermodular } f.}$

For any extreme point solution x to (*), there exists $e \in E$. s.t. $x_e \geq \frac{1}{2}$.

cut requirement function. 2.

$\forall S \subseteq V$,

$$f(S) = \max_{\substack{u \in S \\ v \in \bar{S}}} \{ r(u, v) \}.$$

cut edges.

$\forall S \subseteq V$,

$$\delta(S) = \{ e \in E : e \text{ connects } S, \bar{S} \}.$$

Iterative rounding algorithm

1. $H \leftarrow \phi$. set of edges picked. $u' \leftarrow u$: residual multip limit.
   $f' \leftarrow f$. residual cut requirement.

$\delta_H(S) \equiv$ number of cut edges in H.

2. While $f' \neq 0$.
   ① Solve $(*)$. w.r.t. u'. f'.
   for an extreme point solution x.
   ② For each $e \in E$ with $x_e \geq \frac{1}{2}$.
   - Include $\lceil x_e \rceil$ copies into H.
   - Decrease u_e by $\lceil x_e \rceil$.
   ③ Update f' by $f'(S) \leftarrow f(S) - |\delta_H(S)|$. $\forall S \subseteq V$.
3. Output H.

Solving the LP. – to design a separation oracle.

– Given candidate sol. x'.

define $x_e \equiv x'_e + H_e \quad \forall e.$

since $\delta_x(S) = \delta_{x'}(S) + |\delta_H(S)|$

$f(S) = f'(S) + |\delta_H(S)|.$

$\forall S \subseteq V.$

$\delta_x(S) \geq f(S).$

iff. $\delta_{x'}(S) \geq f'(S).$

Lemma.

$\forall S \subseteq V.$ x' violates cut req. S.

iff. x " " ".

Separation Oracle.

$\forall u, v \in V.$

run minimum u-v cut

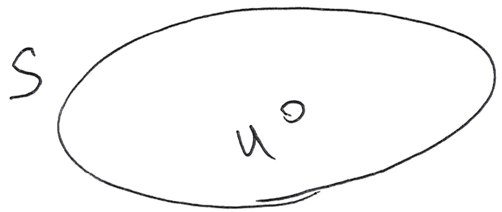
check if it is $\geq r(u, v)$.

Suppose that for some S. $\delta_x(S) < f(S)$.

$\circ\, v$

$f(S) = r(u, v)$

for some $u \in S$. $v \in \bar{S}$.



$\Rightarrow$ minimum s-t cut $< f(S) \leq f(\tilde{S})$.

$\tilde{S}$

*. Theorem 2. The algorithm is a 2-approx. algo.

Consider an iteration in * Step 2.

Let $\hat{x}$ be the multiplicity function of the rounded goals in step 2 ②.

Then. $x - \hat{x}$ is a feasible solution for the new LP.

$$\Rightarrow \text{new OPT} \le \text{cost}(x - \hat{x}).$$

$$\begin{aligned} \Rightarrow \text{Cost}(H + H') &= \text{cost}(H) + \text{Cost}(H') \\ &\le 2 \cdot \text{cost}(\hat{x}) + 2 \cdot \text{(new OPT)} \\ &\le 2 \cdot \text{cost}(\hat{x}) + 2 \cdot \text{cost}(x - \hat{x}) = 2\,\text{cost}(x). \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le 2 \cdot \text{OPT}. \end{aligned}$$

*. It remains to prove Theorem 1.

Def. A function $f: 2^V \to Z^+$ is submodular

if ① $f(V) = 0$.

② $\forall A, B \subseteq V$. the following holds.

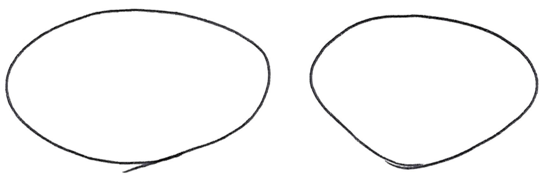
- $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.
- $f(A) + f(B) \geq f(A - B) + f(B - A)$.

functions with diminishing returns.

Lemma. For any graph $G = (V, E)$.
the size of cut edges. $|\delta_G(\cdot)|$ is submodular.

Proof. ① if $A \cap B = \phi$.

then $|\delta_G(A)| + |\delta_G(B)| = |\delta_G(A \cup B)|$.



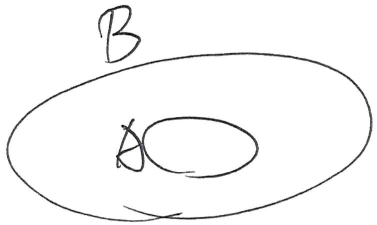
$A \cap B = \phi$. $A - B = A$.
$A \cup B$ $B - A = B$.

② if $A \subseteq B$.

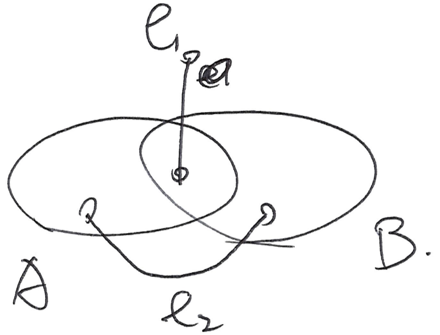
$$\boxed{\begin{array}{l} A \cap B = A, \\ A \cup B = B \end{array}}$$

$A - B = \phi$.

$\cancel{B\,A} =$ $|f(A)| + |f(B)| = |f(B-A)|$.



③



e_1 contributes to $f(A)$. $f(B)$

but not $f(A-B)$ nor $f(B-A)$.

e_2 contributes to $f(A)$. $f(B)$.

but not $f(A \cap B)$ nor $f(A \cup B)$.

$\Rightarrow$. $|f(A)| + |f(B)| \geq |f(A \cap B)| + |f(A \cup B)|$.

Def. A set function $f: 2^V \to \mathbb{Z}^+$ is weakly supermodular

if ① $f(V)=0$.

② $\forall S \subseteq V$, one of the followings holds.

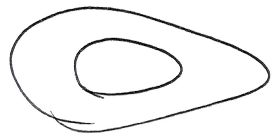
- $f(A)+f(B) \le f(A-B)+f(B-A)$.
- $f(A)+f(B) \le f(A\cap B)+f(A\cup B)$.

Lemma. The cut-requirement function $f(S) = \max\limits_{u\in S,\, v\notin S}(r(u,v))$ is weakly supermodular.

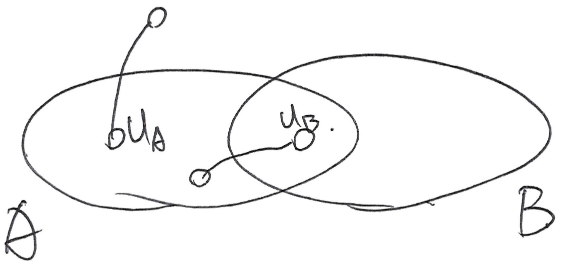
Pf. ① $A\cap B=\phi$.

$\Rightarrow f(A)+f(B) = f(A-B)+f(B-A)$.

② $A\subseteq B$.



$f(A)+f(B) = f(A\cap B)+f(A\cup B)$.

③ 

$\Rightarrow f(A \cap B) = f(B)$.

$f(A \cup B) \geq f(A)$.

etc.

Lemma. The residual cut requirement function f' is weakly supermodular.

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pf. $\forall A, B \subseteq V$. $|\delta_H(A)| + |\delta_H(B)| \geq \begin{cases} |\delta_H(A \cup B)| + |\delta_H(A \cap B)|. \\ |\delta_H(A - B)| + |\delta_H(B - A)|. \end{cases}$

suppose that $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$.

Then. $$\begin{aligned} f'(A) + f'(B) &= f(A) + f(B) - |\delta_H(A)| - |\delta_H(B)|. \\ &\leq (f(A \cap B) - |\delta_H(A \cap B)|) + (f(A \cup B) - |\delta_H(A \cup B)|) \\ &= f'(A \cap B) + f'(A \cup B). \end{aligned}$$

*. Characterization of extreme pt. solutions of (*). for any weakly supermodular function f.

- Let x be an extreme pt. solution with $0 < x_e < 1$. $\forall e \in E$.
- $S \subseteq V$ is tight if $\delta_x(S) = f(S)$.

Theorem. There exists a collection of tight sets $S_1, S_2, \dots, S_m$ such that.

① $A_{S_1}, A_{S_2}, \dots, A_{S_m}$ are linearly independent.

② $S_1, \dots, S_m$ form a laminar family.

$\forall S \subseteq V$. A_S : incidence vector of $\delta(S)$. $\in R^m$.

Def. A set family $L \subseteq 2^V$ is laminar. if $\forall A, B \in L$. either
$$\begin{cases} A \cap B = \phi. \\ A \subseteq B. \\ B \subseteq A. \end{cases}$$