Connectivity Problems with more general Connectivity Requirement.

*. Steiner Forest.

Given. $G=(V,E)$, cost function $c: E \to Q^+$,

collection of disjoint subsets of V, $S_1, S_2, \cdots S_k$.

find min-cost subgraph s.t. S_i is connected for all i.

Define connectivity requirement. $r(u,v) = \begin{cases} 1, & \text{if } u,v \in S_i \text{ for some } i. \\ 0. & \text{otherwise.} \end{cases}$

---

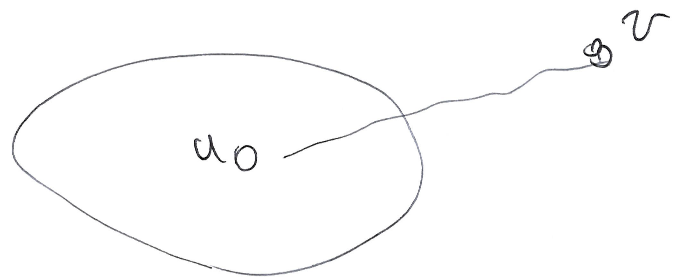
*. ILP formulation. – how?

– Decision: $x_e \in \{0,1\}$ $\forall e \in E$.

But, how should the constraints be written?

(regarding connectivity).

— Observation: for any $S \subseteq V$.



if $\exists\, u \in S,\ v \in \bar{S}$ s.t. $r(u,v) = 1$.

then at least one edge in $\delta(S)$ must be chosen

↑ cut edges of $(S, \bar{S})$

in any feasible solution.

ILP.

$$\min \sum_{e \in E} c_e \cdot x_e .$$

$$\sum_{e \in \delta(S)} x_e \geq f(S) . \quad \forall S \subseteq V .$$

$$x_e \in \{0, 1\} .$$

define $f(S) = \begin{cases} 1. & \text{if } \exists\, u \in S.\ v \in \bar{S}.\ r(u,v) = 1. \\ 0. & \text{otherwise} \end{cases}$

L3

Primal LP.

$$\min \sum_{e\in E} c_e \cdot x_e$$

$$\sum_{e\in\delta(S)} x_e \geq f(S). \quad \forall S\subseteq V.$$

$$x_e \geq 0.$$

Dual LP.

$$\max \sum_{S\subseteq V} f(S)\cdot y_S$$

$$\sum_{S:\, e\in\delta(S)} y_S \leq c_e. \quad \forall e\in E.$$

$$y_S \geq 0. \quad \forall S\subseteq V.$$

- Primal-Dual (Dual-fitting) Algorithm.

  Start with trivial

  $\boxed{\text{Dual}}$ — Improve optimality

  $\boxed{\text{Primal}}$ — Improve feasibility

  Over iterations.

- Raise $\boxed{y_S}$ for some $S\subseteq V$. use it to pay for the primal solution.

Intuition : growing a ball to reach neighboring vertices.

*. Start with singleton vertices. grow balls. until they become connected.

When an edge is tight, i.e.,

$\sum_{S: e \in \delta(S)} y_S \boxed{\leq} c_e$ holds with equality "="

for some e.

the dual values $y_S|_{S: e \in \delta(S)}$ can pay for cost of e.

*. $S \subseteq V$. is "unsatified". if $f(S)=1$. but none of $\delta(S)$ is picked.

*. $S \subseteq V$. is "active" if it is unsatified & minimal in size.

Algorithm

1. $F \leftarrow \phi$. (primal sol)

   $y_S \leftarrow 0$. $\forall S \subseteq V$. (dual sol)

2. While $\exists$ unsatisfied set. do.

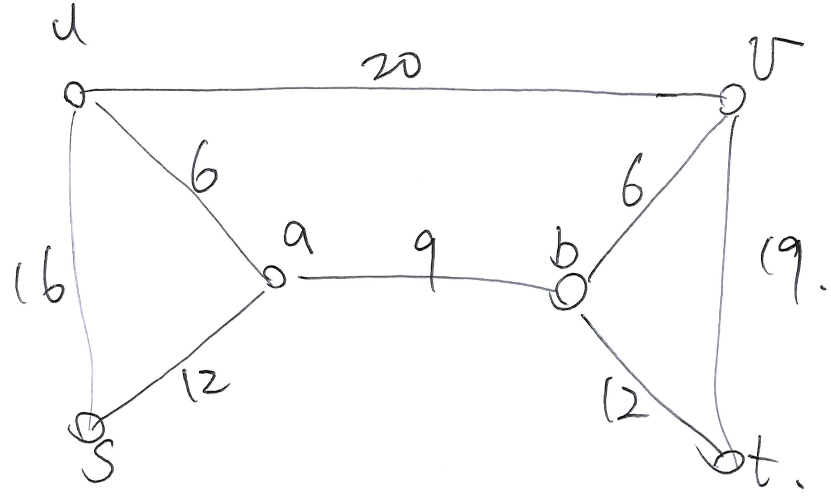
   - Raise y_S for all active $S \subseteq V$. simultaneously until some $e \in \bigcup_{S \text{ active}} \delta(S)$ becomes tight.

   - Pick one such e. set $F \leftarrow F \cup \{e\}$.

3. Let. $F' = \{ e \in F : F - \{e\} \text{ is infeasible} \}$.

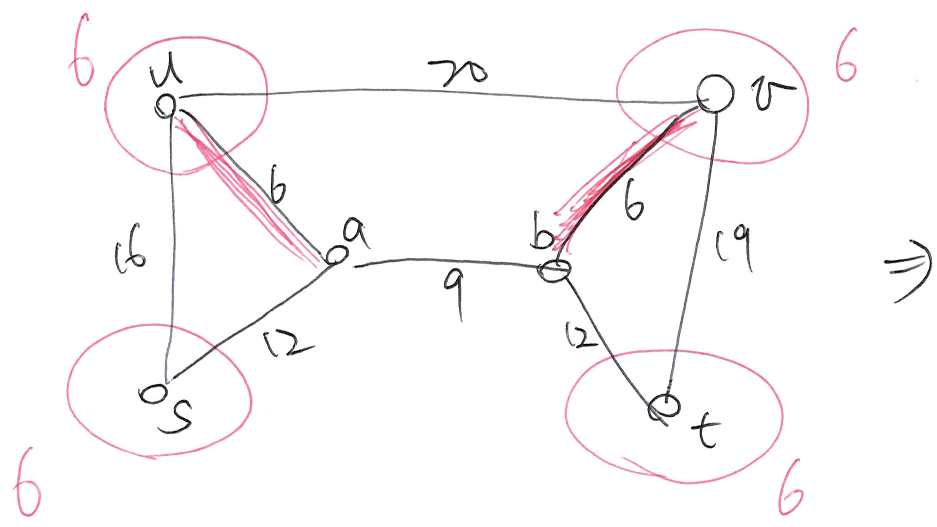
4. Output F'.

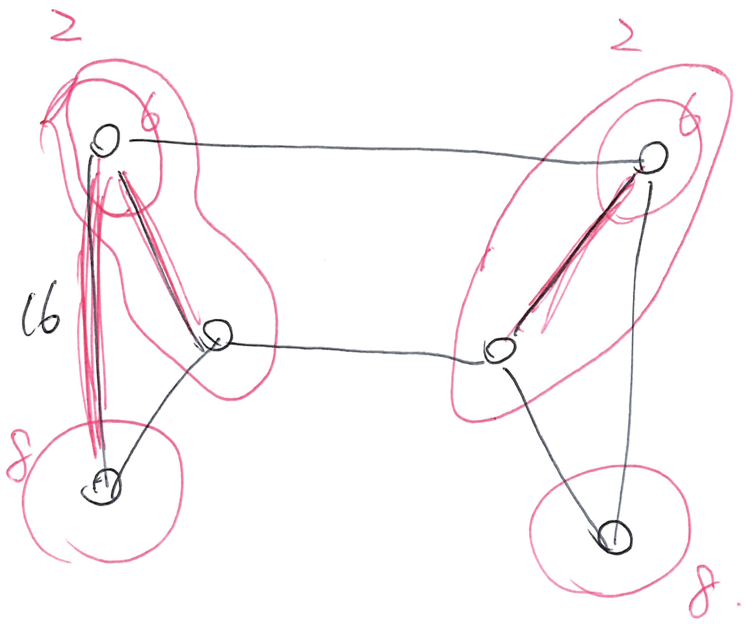
Example.

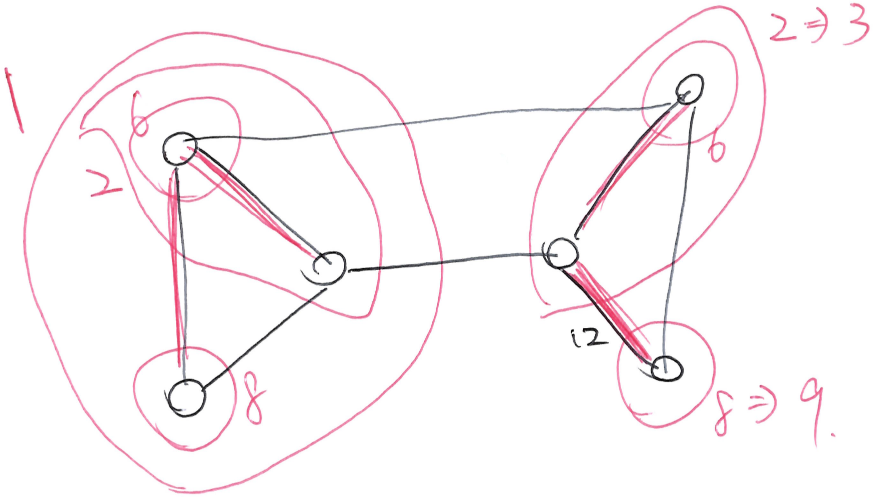


$r(u,v) = r(s,t) = 1.$

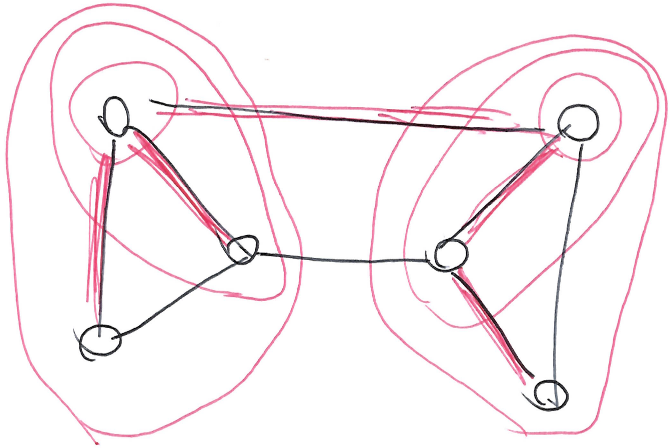
16





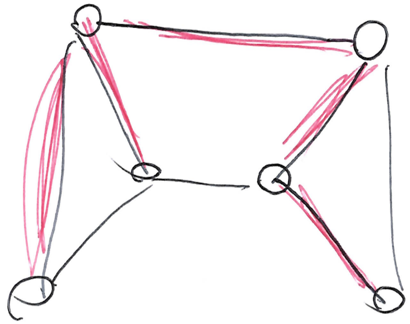


⇒



7

⇓



<u>Analysis.</u>

$$\sum_{e\in F'} c_e = \sum_{e\in F'} \left(\sum_{S:\, e\in\delta(S)} y_S\right).$$

$$= \sum_{S\subseteq V} \left(\sum_{e\in\delta(S)\cap F'} y_S\right) = \sum_{S\subseteq V} \deg_{F'}(S)\cdot y_S.$$

$$\boxed{\overset{?}{\le}}\; 2\cdot\sum_{S\subseteq V} y_S$$

To prove: In each iteration.

$$\Delta\cdot\left(\sum_{S \text{ active}} \deg_{F'}(S)\right) \le 2\cdot\Delta\cdot(\#\text{ of active sets}).$$

$\Rightarrow$ The average degree of active sets w.r.t. F' ~~$\ge$~~ ≤ 2.

The sets form a forest. average degree of any tree ≤ 2.

Solvability of LPs.

- Separation Problem. (for a fixed but unknown polytope Q.)

> Given a point $p \in \mathbb{R}^n$,
> report either — $p \in Q$. ("yes" case)
> — a separating hyperplane for p and Q. ("no" case)

Separation Oracle. — a way to describe the polytope Q.

- (Poly-time) separation Oracle for Q. $\overset{\text{Ellipsoid algorithm.}}{\Longleftrightarrow}$ (Poly-time) $\overset{?}{\exists}\, p \in Q$. (feasibility Problem). $\overset{\text{Binary search.}}{\Longleftrightarrow}$ Weakly (Poly-time) find $p \in Q$ s.t. $c \cdot p$ maxed. (optimization)