There are 5problems, accounting for 100% in total.

Problem 1 (20%). Illustrate the operation of MAX-HEAP-INSERT$(A, 10)$ on the heap $A = \{15, 13, 9, 5, 12, 8, 7, 4, 0, 6, 2, 1\}$.

Problem 2 (20%). We can build a heap by repeatedly calling MAX-HEAP-INSERT to insert the elements into the heap. Consider the following variation on the BUILD-MAX-HEAP procedure:

Algorithm 1 BUILD-MAX-HEAP'

```
A.heap-size = 1
for i = 2 to A.length do
    MAX-HEAP-INSERT(A, A[i])
end for
```

1. Do the procedure BUILD-MAX-HEAP and BUILD-MAX-HEAP' always create the same heap when run on the same input array? Prove that they do, or provide a counterexample.
2. Show that in the worst case, BUILD-MAX-HEAP' requires $\Theta(n \log n)$ time to build an n-element heap.

Problem 3 (20%). Argue that any comparison-based algorithm for constructing a binary search tree from an arbitrary list of n elements takes $\Omega(n \log n)$ time in the worst case.

Problem 4 (20%). Show that the longest simple path from a node x in a red-black tree to a descendant leaf has length at most twice that of the shortest simple path from node x to a descendant leaf.

Problem 5 (20%). (Vandermonde matrix.)
Given numbers $x_0, x_1, \ldots, x_{n-1}$, prove that the determinant of the Vandermonde matrix

$$V(x_0, \ldots, x_{n-1}) = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix}$$

is

$$\det(V(x_0, \ldots, x_{n-1})) = \prod_{0 \le j < k \le n-1} (x_k - x_j).$$

Hint: Multiply column i by $-x_0$ and add it to column $i+1$ for $i = n-1, n-2, \ldots, 1$, and then use induction.