# Introduction to Algorithms

Mong-Jen Kao (高孟駿)

Tuesday 10:10 – 12:00

Thursday 15:30 – 16:20

# Data Structures

Particular ways of storing data *to support special operations*.

# Min- (Max-) Heap / Priority Queue

Storing semi-dynamic data to extract the minimum element fast.

# Priority Queue

- Suppose that we want to maintain a set $A$ of *elements****, each associated with a key,*** so as to support the following operations.
  - **Insert(A, x)** – to insert a given element $x$ into $A$.
  - **Maximum(A)** – to return the largest element in $A$.
  - **Extract-Max(A)** – to remove and return the largest element from $A$.
  - **Increase-Key(A, x, k)**
    – to increase the value of the elements $x$'s key to the new value $k$.

# Priority Queue

With ***max-heap***,
these operations can be done in…

- Suppose that we want to maintain a set $A$ of *elements*, ***each associated with a key,*** so as to support the following operations.
  - **Insert(A, x)** – to insert a given element $x$ into $A$.
  - **Maximum(A)** – to return the largest element in $A$.
  - **Extract-Max(A)** – to remove and return the largest element from $A$.
  - **Increase-Key(A, x, k)**
    – to increase the value of the elements $x$'s key to the new value $k$.

$O(\log n)$ time.

$O(1)$ time.

$O(\log n)$ time.

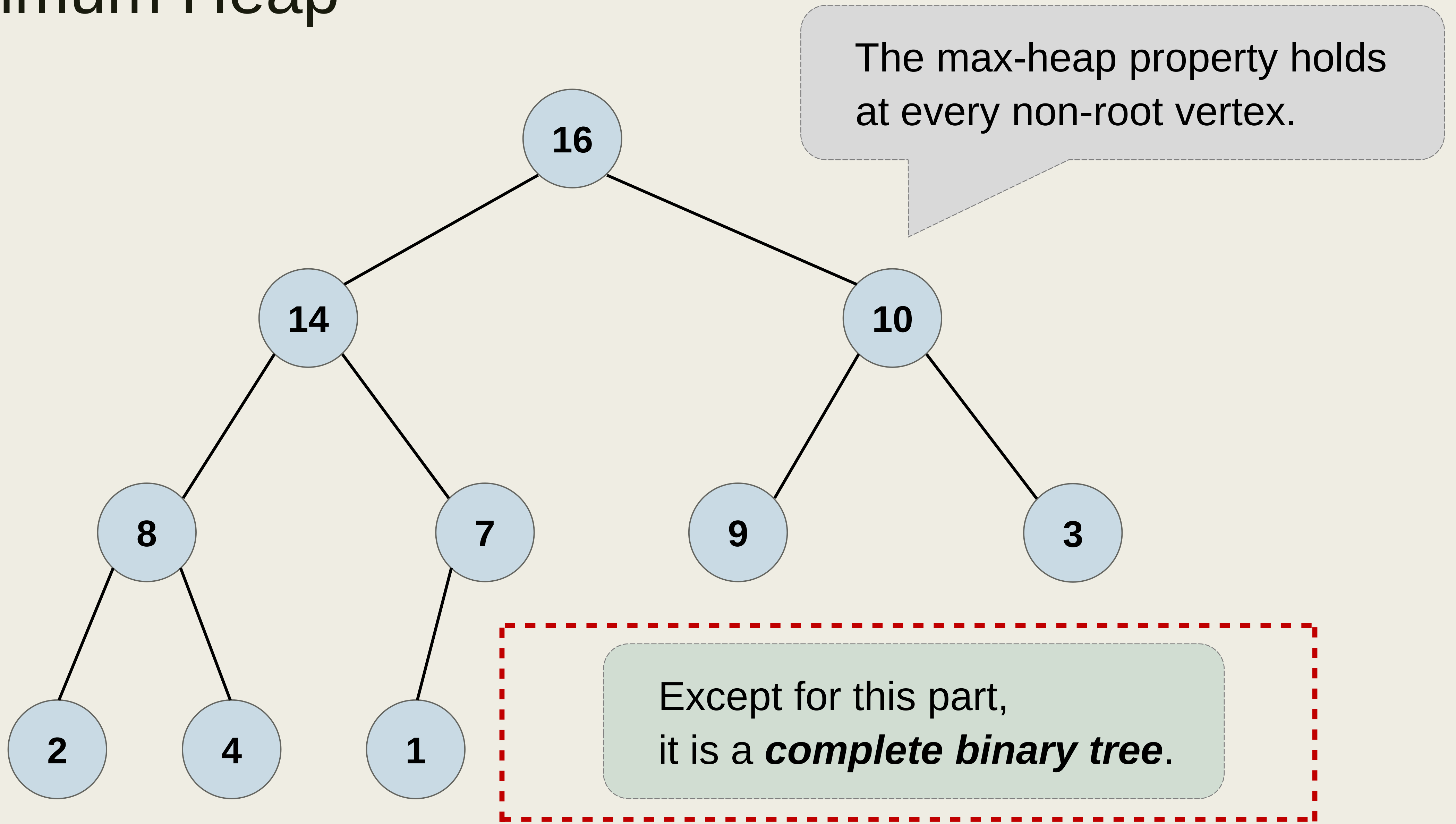
# Maximum Heap

- The maximum heap is a ***nearly complete binary tree*** such that

  - *The* ***nodes*** in the tree are ***comparable*** *to each other*.

  - (*Max-Heap property*)
    For any non-root node $v$ and its parent $p(v)$,
    we always have
    $$p(v) \geq v \ .$$

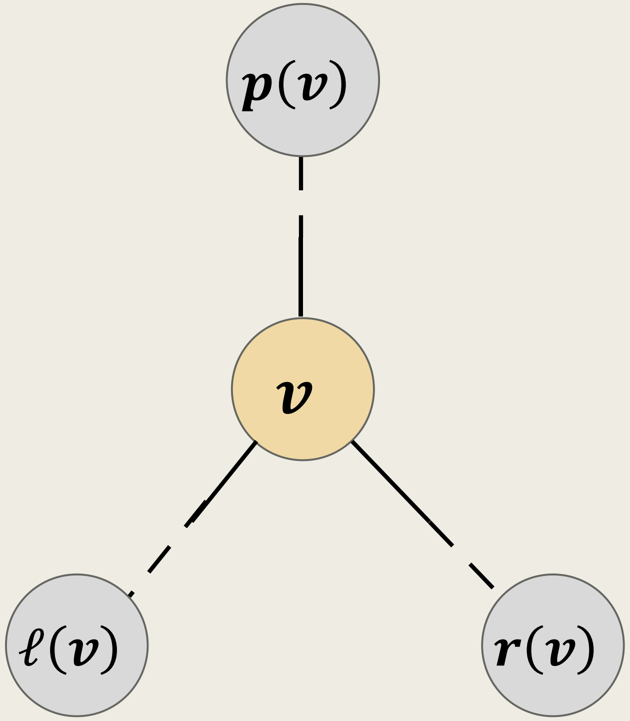
# Maximum Heap



# Representing a Binary Tree

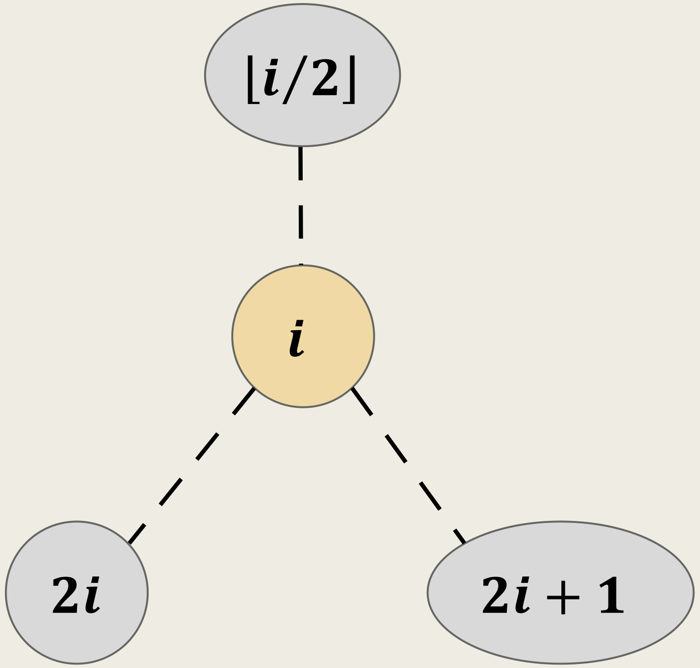
- **In general**, to record the structure of a binary tree $T = (V, E)$, for each node $v \in V$, we need to store the following information.
  - The parent node of $v$, denoted $p(v)$.
  - The left- and right- children nodes of $v$, denoted $\ell(v)$ and $r(v)$, respectively.

```
struct node {
    int val;
    node *p, *l, *r;
};
```

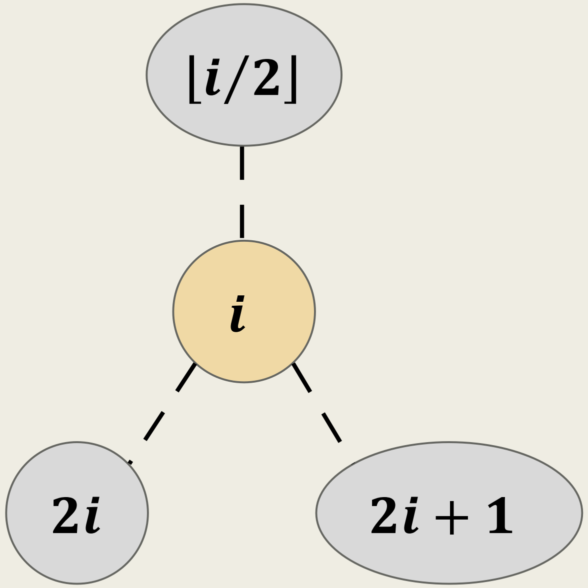


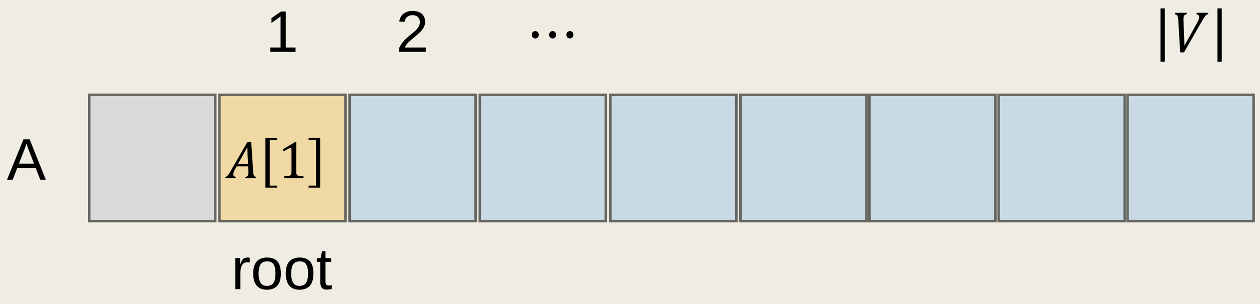
# Representing a **Nearly-Complete** Binary Tree

- For a nearly-complete binary tree $T = (V, E)$, we can use **an array $A$ of size $O(|V|)$** to represent it.
  - The root is $A[1]$.
  - Given an index $i \geq 1$,
    - Parent$(i) \coloneqq \lfloor i/2 \rfloor$.
    - Left$(i) \coloneqq 2i$ .
    - Right$(i) \coloneqq 2i + 1$ .



- For a nearly-complete binary tree $T = (V, E)$, we can use ***an array $A$ of size $O(|V|)$*** to represent it.
  - The root is $A[1]$.
  - Given an index $i \geq 1$,
    - Parent($i$) $\coloneqq \lfloor i/2 \rfloor$.
    - Left($i$) $\coloneqq 2i$ .
    - Right($i$) $\coloneqq 2i + 1$ .



# Properties of Array-Representation

- Let $A$ be an array representation of a nearly complete binary tree $T = (V, E)$ and let $\boldsymbol{n} = |\boldsymbol{V}|$. We have the following properties.
  - Each of the nodes at

    $$\lfloor n/2 \rfloor + 1, \qquad \lfloor n/2 \rfloor + 2, \qquad \cdots\cdots, \qquad n$$

    is a leaf node.
  - For any $1 \le h \le \lfloor \log n \rfloor + 1$, there are at most

    $$\lceil n/2^h \rceil$$

    nodes at height $h$.

In the following,
we assume array representation.

# Maintain the Heap Property

- We introduce a procedure for maintaining a max-heap.
- The **Max-Heapify**$(\boldsymbol{A}, \boldsymbol{i})$ procedure takes as input
  - A nearly complete binary tree $T$ with root $i$, where
  - Both of **Left**$(\boldsymbol{i})$ **and Right**$(\boldsymbol{i})$, if not empty, **are both <u>max-heaps</u>**.
- The Max-Heapify procedure guarantees that $T$ is a max-heap after execution in $O(\log|T|)$ time.

- Max-Heapify$(A, i)$

  -- To assure the heap property for the tree rooted at $i$.

  -- Assumption: Left$(i)$ and Right$(i)$, if not empty, are max-heaps.

---

A. Let $k \coloneqq i$.

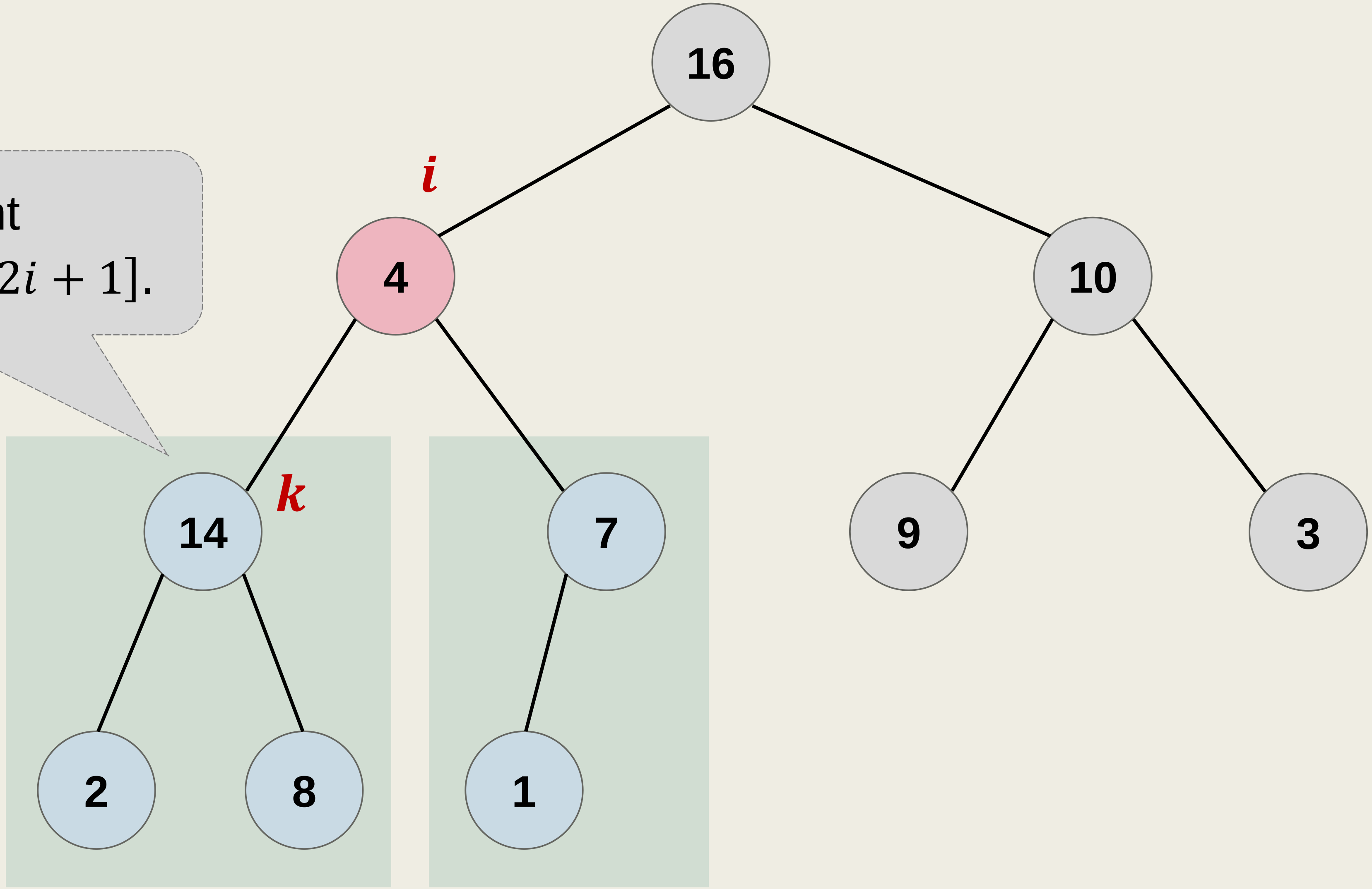
B. If $2i \leq heap\_size[A]$ and $A[2i] > A[k]$, then $k \coloneqq 2i$.

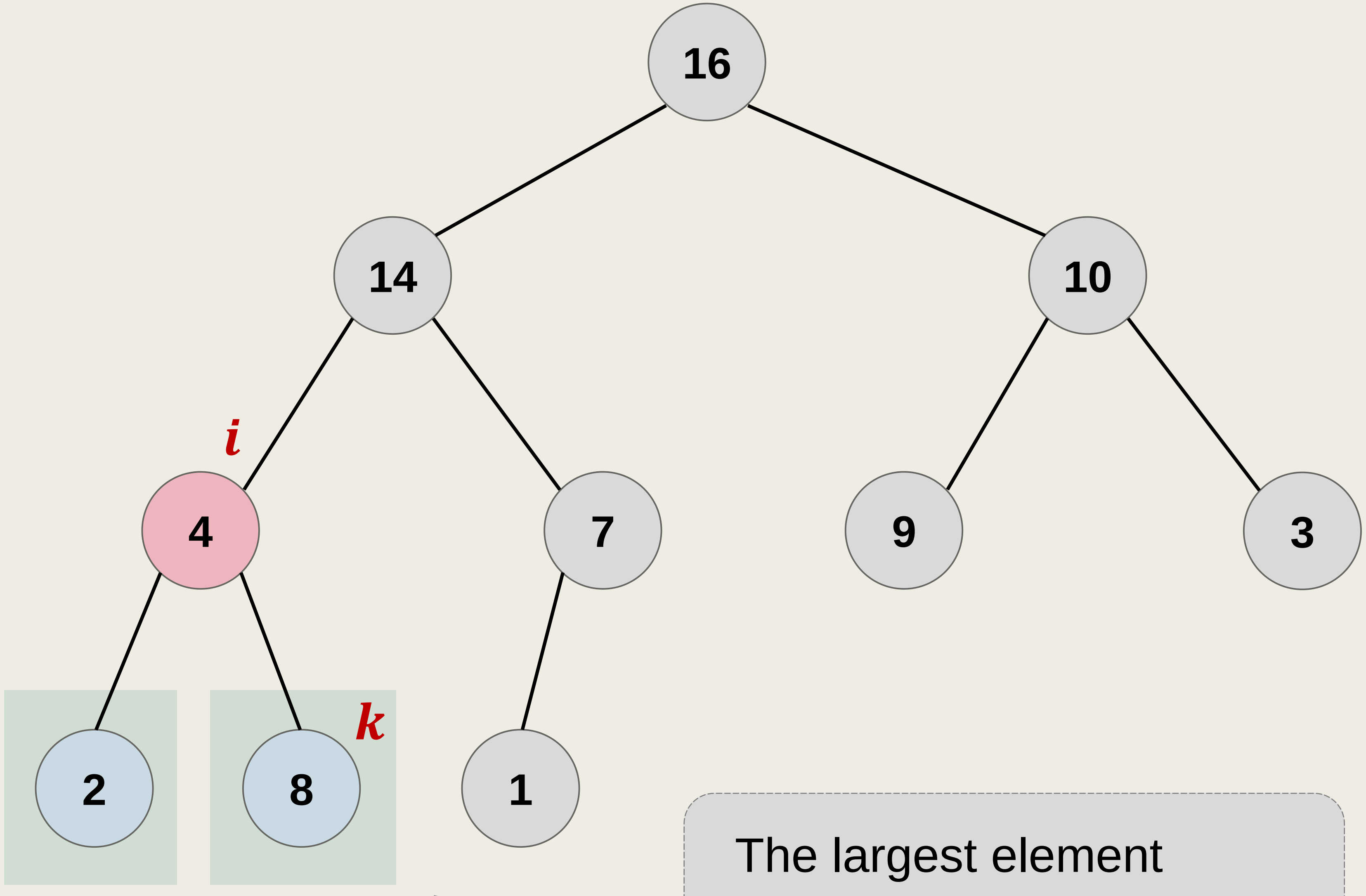
   If $2i + 1 \leq heap\_size[A]$ and $A[2i + 1] > A[k]$, then $k \coloneqq 2i + 1$.

C. If $k \neq i$, then

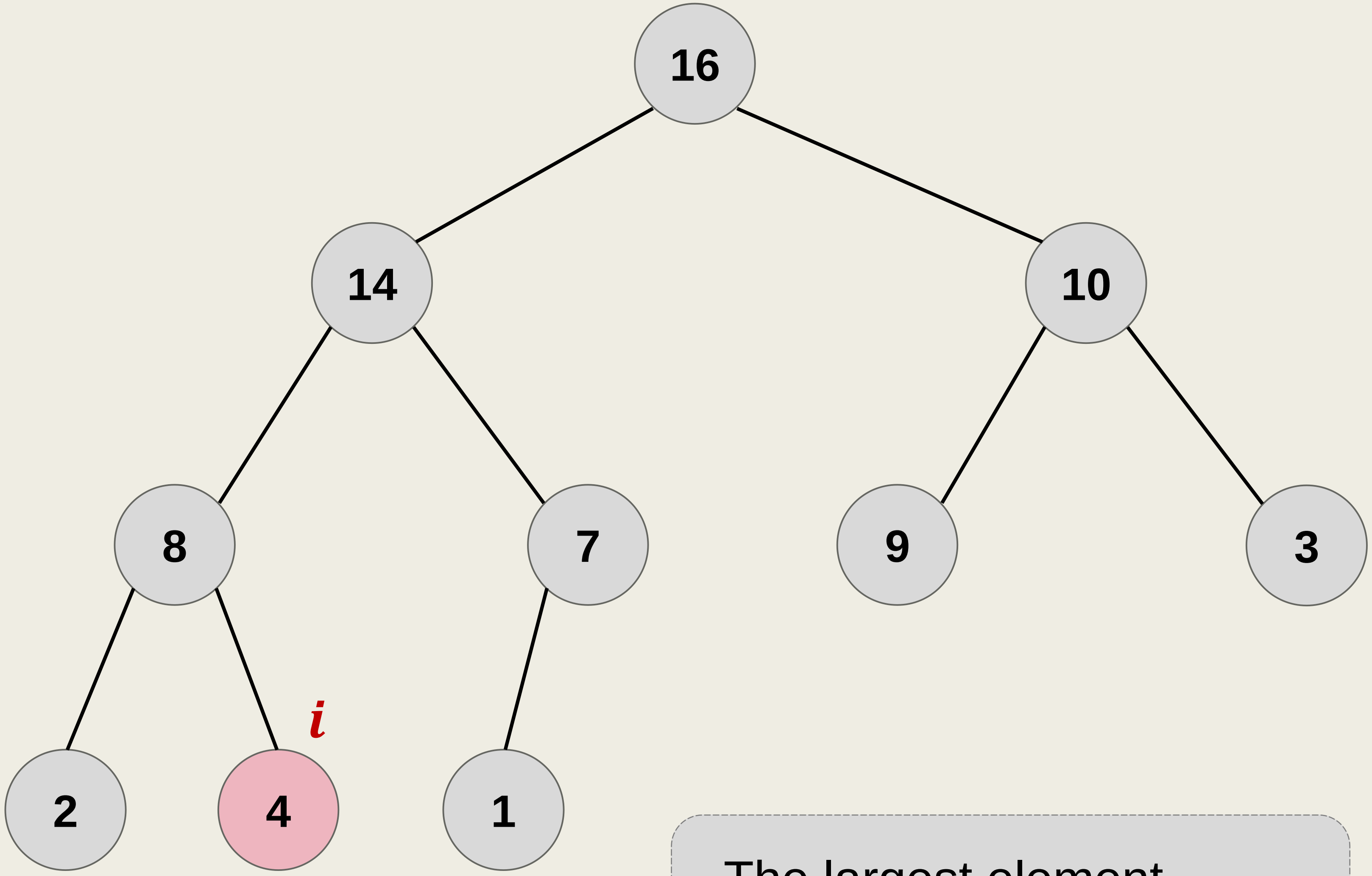
   - Exchange $A[i]$ with $A[k]$.
   - Max-Heapify$(A, k)$.

The largest element
from $A[i]$, $A[2i]$, $A[2i+1]$.

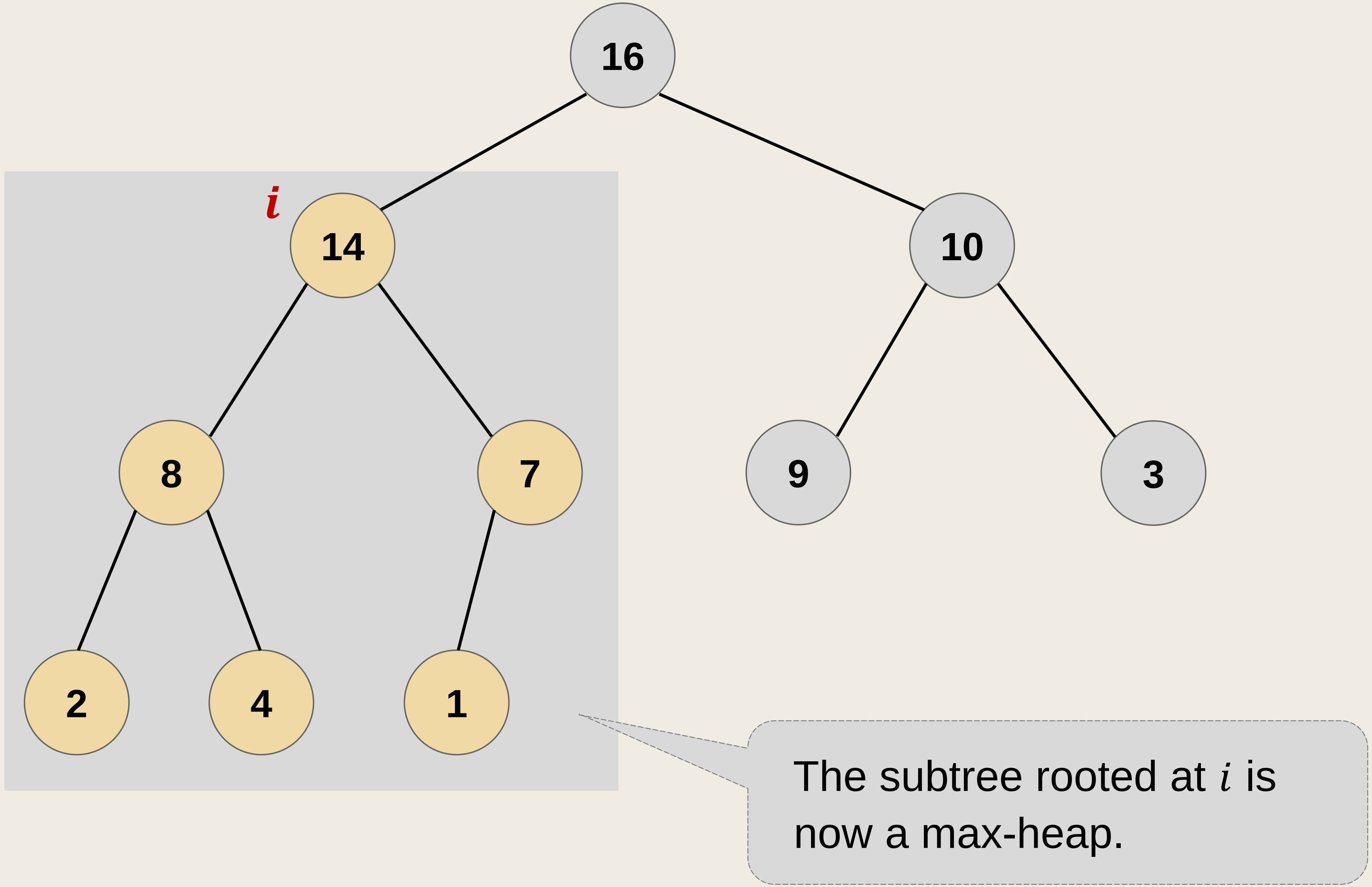





The largest element
from $A[i]$, $A[2i]$, $A[2i+1]$.




The largest element from $A[i]$, $A[2i]$, $A[2i+1]$.


16
i
14
10
8
7
9
3
2
4
1
The subtree rooted at $i$ is now a max-heap.

# Building a Heap in $O(n)$ Time

# Building the Heap in $O(n)$ Time

- The Build-Max-Heap$(A)$ procedure takes an array $A$ as input and ***builds a max-heap*** for the elements in $A$ ***in place***.
  - This procedure proceeds in a bottom-up manner and uses the Max-Heapify procedure to guarantee the heap property.

---

- Build-Max-Heap$(A)$

A. $heap\_size[A] \coloneqq \mathrm{length}[A]$.

B. for $i = \mathrm{length}[A]$ down to $1$, do

Max-Heapify$(A, i)$.

# Analysis of Build-Max-Heap

- Recall that,

  the call to Max-Heapify on an element at height $h$ takes $O(h)$ time.

- For any $1 \leq h \leq \lfloor \log n \rfloor + 1$, there are at most $\lceil n/2^h \rceil$ nodes at height $h$.

- Hence, the total running time of Build-Max-Heap is

$$\sum_{1 \leq h \leq \lfloor \log n \rfloor + 1} \left\lceil \frac{n}{2^h} \right\rceil \cdot O(h) \; = \; O\left( n \cdot \sum_{h \geq 0} \frac{h}{2^h} \right).$$

# Analysis of Build-Max-Heap

- To bound $\sum_{h \geq 0} h/2^h$, observe that

$$\sum_{i \geq 0} x^i = \frac{1}{1-x}$$

  holds for all $x$ with $|x| < 1$.

- Differentiating both sides of the equation on $x$, we obtain that

$$\sum_{i \geq 1} i \cdot x^{i-1} = \frac{1}{(1-x)^2} \quad \text{holds for any } |x| < 1.$$

# Analysis of Build-Max-Heap

- Differentiating both sides of the equation w.r.t. $x$, we obtain that

$$\sum_{i \geq 1} i \cdot x^{i-1} = \frac{1}{(1-x)^2} \quad \text{holds for any } |x| < 1.$$

- Taking $x = 1/2$, we obtain that

$$\sum_{h \geq 0} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2.$$

- Hence,

$$\sum_{1 \leq h \leq \lfloor \log n \rfloor + 1} \left\lceil \frac{n}{2^h} \right\rceil \cdot O(h) = O(n) \cdot \sum_{h \geq 0} \frac{h}{2^h} = O(n).$$

# Extracting the Maximum Element

# Extracting the Maximum Element

- To extract the maximum element from a max-heap $A$, we swap the root with the last element, and perform Max-Heapify.
  - The time it takes is $O(\log n)$.

- Extract-Max$(A)$

---

A. Exchange $A[1]$ with $A\left[heap\_size[A]\right]$.

B. Decrease $heap\_size[A]$ by 1 and call Max-Heapify$(A, 1)$.

C. Return $A[heap\_size[A] + 1]$.

# The Heapsort Algorithm

# Heapsort

- With the procedure we have so far,
  we can do sorting in $O(n \log n)$ time with max-heap.

---

- Heapsort$(A)$

A. Build-Max-Heap$(A)$.

B. For $i = \mathrm{length}[A]$ down to $2$, do

Extract-Max$(A)$.

# Other Operations

# Increase the Value of an Element

- We can change the value of an element.  After that, we need to ensure the heap property. Overall it takes $O(\log n)$ time.
  - Perform Max-Heapify if the value is decreased.
  - Otherwise, we proceed upward if the value is increased.

---

- Max-Heap-Increase-Key$(A, i, key)$ -- Assumption: $key > A[i]$.

A. $A[i] \leftarrow key$.

B. While $i > 1$ and $A[i/2] < A[i]$, do
   - Exchange $A[i]$ with $A[i/2]$ and set $i \leftarrow i/2$.

# Insert a new Element

- To insert an element, we insert it at the end of the heap and perform the increase-key operation.
  - The time it takes is $O(\log n)$.

- Max-Heap-Insert$(A, key)$

---

A. Increase $heap\_size[A]$ by 1.

B. Call Max-Heap-Increase-Key$(A, heap_size[A], key)$.

# Priority Queues

# Priority Queue

With ***<u>max-heap</u>***,
these operations can be done in…

■ Suppose that we want to maintain a set $A$ of ***elements, each associated with a <u>key</u>,*** so as to support the following operations.

– **Insert(A, x)** – to insert a given element $x$ into $A$.

$O(\log n)$ time.

– **Maximum(A)** – to return the largest element in $A$.

$O(1)$ time.

– **Extract-Max(A)** – to remove and return the largest element from $A$.

– **Increase-Key(A, x, k)**
– to increase the value of the elements $x$'s key to the new value $k$.

$O(\log n)$ time.

# Mergeable Heaps – A Note

# Mergeable Heaps

- Mergeable Heaps refer to the data structures that supports the following operations.
  - Make_Heap() – to create and return an empty heap.
  - Insert(H, x) – to insert a given element $x$ into $H$.
  - Minimum(H) – to return the smallest element in $H$.
  - Extract-Min(H) – to remove and return the smallest element from $H$.
  - **Union($\boldsymbol{H_1}$, $\boldsymbol{H_2}$)** – to create and return the union of $H_1$ and $H_2$. The heaps $H_1$ and $H_2$ are destroyed by this operation.

# Mergeable Heaps

- This type of structures often supports the following two operations as well.
  - Decrease-Key($H, x, k$) – to assign the element $x$ a smaller key $k$.
  - Delete (H, x) – to delete a given node $x$ from $H$.

# Mergeable Heaps

| Procedure | Binary Heap (worst-case) | Binomial Heap (worst-case) | Fibonacci Heap (amortized/average) |
|---|---|---|---|
| Make-Heap | $\Theta(1)$ | $\Theta(1)$ | $\Theta(1)$ |
| Insert | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(1)$ |
| Minimum | $\Theta(1)$ | $\Theta(\log n)$ | $\Theta(1)$ |
| Extract-Min | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(\log n)$ |
| Union | $\Theta(n)$ | $\Theta(\log n)$ | $\Theta(1)$ |
| Decrease-Key | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(1)$ |
| Delete | $\Theta(\log n)$ | $\Theta(\log n)$ | $\Theta(\log n)$ |

As the semesters are shortened,
we may not be able to examine them in this semester.