Efficient computer analysis of switched capacitor circuits via the z-domain.

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Outline

In a previous lecture time-,frequency- and z-domain equations of switched capacitor circuits have been derived on an intuitive and algebraic manner.

In this lecture we continue with the computer implementation aspects of the z-domain analysis technique in order to compute the frequency, aliasing and sensitivity properties of switched capacitor circuits (S.C.).

The setup, the compaction and the solution of the z-domain equations are described. The advantages and use of the adjoint switched capacitor network are motivated for the computation of sensitivity characteristics.

After the problem statement (section 1), the derivation of the most important equations is given (section 2). The next sections (3,4,5) show how these calculations can be efficiently performed in a computer analyses. Some practical applications demonstrate the use of the analysis as implemented in the DIANA-program.

1. Problem statement.

1.1. Introduction.

The analysis of switched capacitor networks is, except for trivial cases very complicated. Even by use of a computer the rather straightforward time domain method of the DIANA program often results in rather long calculation times. This motivated the direct computation of frequency, aliasing and sensitivity properties of general periodically switched capacitor networks in the DIANA program. Therefore a set of linear equations, the size of which is proportional to the number of phases (N) and the size of the network (s) is used. By making full use of the matrix structure, a very efficient matrix compaction algorithm is obtained which is linear in the number of phases. With the appropriate choises of the righthand side and the use of the adjoint switched capacitor network the number of linear equations to be solved are further minimized.

Because digital filters are sampled data systems as well as SC filters, most of the simulation tools for SC filters can easily be extended towards digital filter simulation.

1.2. SC-circuits.

We define SC circuits to be arbitrary linear networks containing ideal switches, capacitors, independant voltage and charge sources and dependant sources VCVS, QCQS, QCVS and VCQS. The switches are controlled by Boolean clock variables $\phi_i(t) = 0$ or 1. $\phi_i(t)=0$ (resp. $\phi_i(t) = 1$) corresponds to an open (resp. closed) switch at time t if this switch is driven by clock i. The time is partitioned into time slots $\Delta_k=(t_k,t_{k+1}]$ such that the clock signals do not vary in Δ_k, i.e. $\phi_i(t) = \phi_{ik}$ for $t \in \Delta_k$. We assume that the clock signals are T-periodic, with N time slots (called N phases) in one period of duration T.

1.3. S.C.-analysis problem.

For the analysis of a SC-circuit in the frequency domain one or more input voltage sources are applied in certain branches as an excitation to the network. As a result of the analysis one obtains the voltage responses at one or more nodes at a set of pulsations $\Omega = \{\omega_1, \omega_2 \ldots \omega_i\}$ in the base band $[0,\omega_s = 2\pi/T]$ or in higher or lower bands.

If we apply a sinusoidal voltage excitation in a certain branch j we normally obtain a steady state voltage response which is not purely sinusoidal. Therefore we are mainly interested in $H^{(ij)}(\omega_m)$, $\omega_m \in \Omega$ which is the transferfunction between a sinusoidal voltage excitation in branch j at ω_m and the component at the same pulsation in the steady state of the voltage response at node i. It should include the input-output effects of the continuous coupling.

Sinusoidal excitations at $\omega_m+n\omega_s$ in the higher bands than the base band normally have components in the steady state response inside the baseband at pulsation ω_m. Therefore one is interested in the aliasing term of the transferfunction $X^{(ij)}(\omega_m,\omega_m+n.\omega_s)$ due to the effect of a sinusoidal voltage excitation in branch j at pulsation $\omega_m+n\omega_s$ on the steady state voltage response at node i at pulsation ω_m. This allows to setup and check the anti-aliasing requirements.

In sensitivity analysis the relative transferfunction variation caused by a relative element variation is examined :

$$S^{H^{(ij)}}(\omega_m) \triangleq \frac{\mu}{H^{(ij)}} \cdot \frac{\partial H^{(ij)}(\omega_m)}{\partial \mu} \quad ,\omega_m \in \Omega \qquad (1)$$

Where μ can be a capacitor value or an amplification factor of a dependent source.

The basic equations which solve these problems have been obtained by (1° and 3° Vandewalle, 1981) and (3° De Man, 1980) and are presented in a previous lecture.

2. Derivation of the equations for the computer aided analysis.

Next we summarize the expressions obtained in (1°,3° Vandewalle, 1981) in order to solve the problems of frequency domain-, aliasing- and sensitivity analysis.
Later on we show how these calculations can be efficiently performed.

2.1. Motivation.

In a previous lecture the tableau matrix approach was used as a general vehicle for the derivation of the z-domain analysis matrix. This tableau matrix has the property that it is very general and that all network variables like nodevoltages, branchvoltages and branch charges can be obtained from it.

As already mentioned there, other approaches can be derived from the tableau-formulation.

The modified nodal analysis approach (MNA) by (Ho, 1975) is a matrix formulation that is very well suited for computer implementation because it can be setup from topological network data. The number of equations is much smaller than in the tableau matrix and a little higher than for the nodal matrix. However it does not have the drawbacks of difficult processing of voltage sources and charge dependent circuit elements as is the case in the nodal matrix. In fact the MNA method is used by most modern circuit analysis programs as e.g. SPICE2 and DIANA.

The MNA -framework of SC-circuits in the time domain was introduced by (3• De Man, 1980). This was done by using the existing circuit analysis program DIANA and putting the timestep h=1 with a backward Euler integration rule which results in the MNA-time domain equations for SC-circuits. The use of one calculation per clockvariation resulted in an efficient timedomain analysis.

The MNA-framework has been extended towards the z-domain analysis (3• Vandewalle, 1981).

2.2. Example of the setup of the MNA equations.

In the formulation of the MNA circuit equations, the node voltages with respect to a common datum node (usually the ground node) are taken as variables $\underset{\sim}{V}_k$ in each timeslot k. The law of charge conservation is applied to each node in each timeslot. Branch charges $\underset{\sim}{Q}_k$ trough voltage sources, two node switches, VDV's, QDV's and QDQ's and the controlling branches of QDQ's are taken as additional variables, together with the corresponding branch constitutive relations which deliver additional equations. These additional variables are available as output variables or are necessary e.g. for sensitivity analysis as will be shown later.

Example : Consider as an example a first order low pass filter (Caves, 1977) with its corresponding clocking sequence shown in fig. 1.

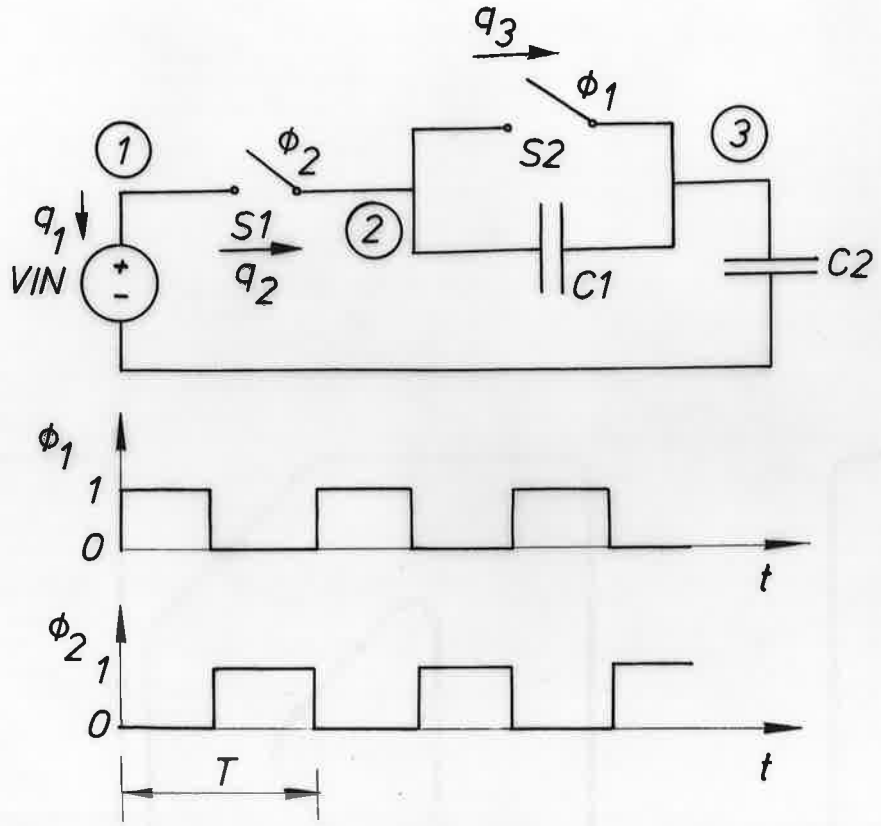


Fig. 1. First order low pass filter.

The MNA equations are given by :

- Timeslot 1 :

node 1 : $Q_{11} = 0$

node 2 : $C_1 (V_{21}-V_{31})+Q_{31} = z^{-1}C_1 (V_{22}-V_{32})$

node 3 : $C_1 (V_{31}-V_{21})+C_2 V_{31} -Q_{31} = z^{-1}C_1(V_{32}-V_{22}) + z^{-1}C_2V_{32}$

branch V_{IN}: $V_{11} = V_{i1}$

branch S1 : $Q_{21} = 0$

branch S2 : $V_{21}-V_{31} = 0$

- Timeslot 2 :

node 1 : $Q_{12} + Q_{22} = 0$

node 2 : $C_1(V_{22}-V_{32}) - Q_{22} = C_1(V_{21}-V_{31})$

node 3 : $C_1(V_{32}-V_{22}) + C_2V_{32} = C_1(V_{31}-V_{21}) + C_2V_{31}$

branch V_{IN} : $V_{12} = V_{i2}$

branch S1 : $V_{12} - V_{22} = 0$

branch S2 : $Q_{32} = 0$

The z-domain MNA equations can be put together in the two phase z-domain MNA matrix as follows in (2) and (7).

$$\left[\begin{array}{cc|cc} \underset{\sim}{A}_1 & \underset{\sim}{B}_1 & -\underset{\sim}{E}_1 z^{-1} & \underset{\sim}{0} \\ \underset{\sim}{C}_1 & \underset{\sim}{D}_1 & \underset{\sim}{0} & \underset{\sim}{0} \\ \hline -\underset{\sim}{E}_2 & \underset{\sim}{0} & \underset{\sim}{A}_2 & \underset{\sim}{B}_2 \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{C}_2 & \underset{\sim}{D}_2 \end{array}\right] \cdot \left[\begin{array}{c} \underset{\sim}{V}_1 \\ \underset{\sim}{Q}_1 \\ \hline \underset{\sim}{V}_2 \\ \underset{\sim}{Q}_2 \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{W}_1 \\ \underset{\sim}{U}_1 \\ \hline \underset{\sim}{W}_2 \\ \underset{\sim}{U}_2 \end{array}\right] \qquad (2)$$

2.3. General z-domain MNA equations.

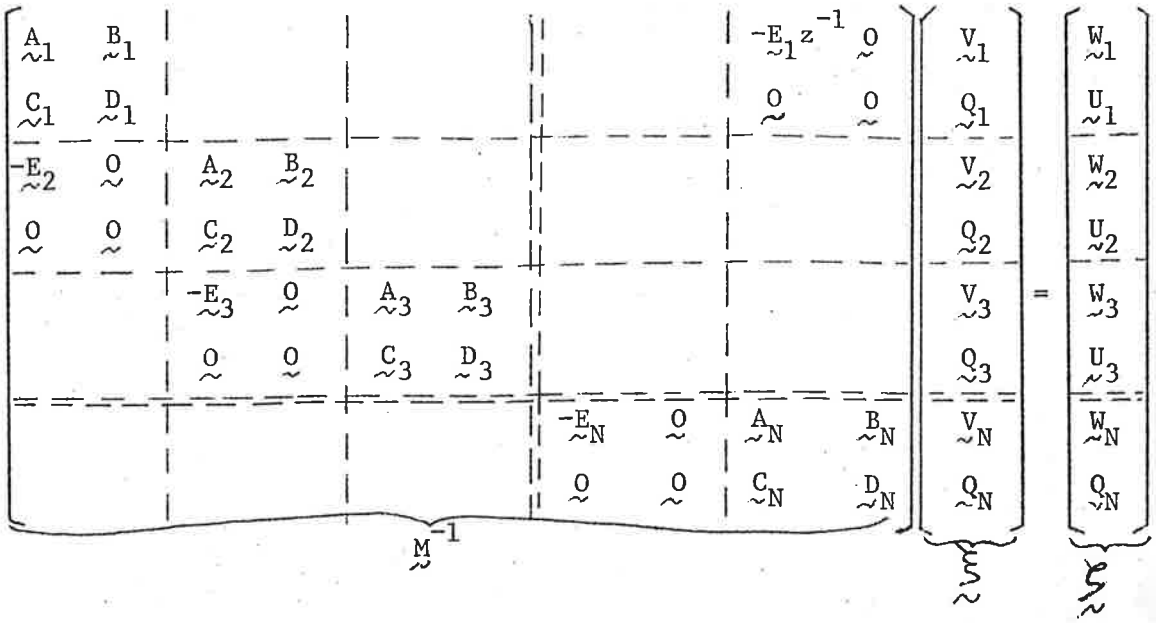
We can define the MNA equations more generally in the z-domain MNA formalism. (3• Vandewalle, 1981).

Let $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3 \cdots \underset{\sim}{v}_N, \underset{\sim}{v}_{N+1} \cdots\}$ (resp., $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2 \ldots\}$) be the piece-wise-constant voltage responses at the nodes (resp., voltages sources in some selected branches). In other words $\underset{\sim}{v}_\ell$ is the node voltage vector in Δ_ℓ. Also let $\{\underset{\sim}{q}_1, \underset{\sim}{q}_2 \ldots\}$ (resp., $\{\underset{\sim}{w}_1, \underset{\sim}{w}_2, \ldots\}$) be the charges transferred in some selected branches (resp., injected by charge sources in the nodes). More precisely $\underset{\sim}{q}_\ell$ is the charge vector transferred in these branches between the end t_ℓ^- of the time slot $\Delta_{\ell-1}$ and $t_{\ell+1}^+$. Each of these sequences is partitioned into N different sequences one for each phase and z-transformed e.g.

$$\underset{\sim}{V}_k(z) \triangleq \mathcal{Z}\left\{\underset{\sim}{v}_{k+\ell N}\right\} = \sum_{\ell=0}^{\infty} \underset{\sim}{v}_{k+\ell N} z^{-\ell}, \quad k = 1, 2, \ldots N \qquad (3)$$

Then it is proven in (3• Vandewalle, 1981) that the input z-transforms $\underset{\sim}{W}_k$, $\underset{\sim}{U}_k$, k = 1, ... N are related to the output z-transforms $\underset{\sim}{V}_k$, $\underset{\sim}{Q}_k$, k = 1, ... N by :

$$
\begin{array}{c|ccc:c:ccc:ccc:ccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 1' & 2' & 3' & 4' & 5' & 6' \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & C_1 & -C_1 & 0 & 0 & 1 & 0 & -C_1 z^{-1} & C_1 z^{-1} & 0 & 0 & 0 \\
3 & 0 & -C_1 & C_1+C_2 & 0 & 0 & -1 & 0 & C_1 z^{-1} & (-C_1-C_2) z^{-1} & 0 & 0 & 0 \\
\hline
\# 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
2' & 0 & -C_1 & C_1 & 0 & 0 & 0 & 0 & C_1 & -C_1 & 0 & -1 & 0 \\
3' & 0 & C_1 & -C_1-C_2 & 0 & 0 & 0 & 0 & -C_1 & C_1+C_2 & 0 & 0 & 0 \\
\hline
\# 4' & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
5' & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
6' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
\cdot
\begin{array}{c}
V_{11} \\ V_{21} \\ V_{31} \\ \hline Q_{11} \\ Q_{21} \\ Q_{31} \\ \hline V_{12} \\ V_{22} \\ V_{32} \\ \hline Q_{12} \\ Q_{22} \\ Q_{32}
\end{array}
=
\begin{array}{c}
0 \\ 0 \\ 0 \\ \hline V_{i1} \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline V_{i2} \\ 0 \\ 0
\end{array}
\qquad (7)
$$



(4)

where the missing entries are zero and where the matrices $\underset{\sim}{A}_k$, $\underset{\sim}{B}_k$, $\underset{\sim}{C}_k$, $\underset{\sim}{D}_k$ and $\underset{\sim}{E}_k$ characterize the SC network.

The equations (4) contain N blocks, one for each phase. In each block the first part consists of node charge equations and the second part are the constitutive equations of voltage sources, dependent sources and switches.

Let $\underset{\sim}{M}$ be the inverse of the matrix in (4), call it the z-domain transfer matrix and partition it accordingly into

$$\underset{\sim}{M} = \left[\begin{array}{cc|cc|c|cc} \underset{\sim}{G}_{11} & \underset{\sim}{H}_{11} & \underset{\sim}{G}_{12} & \underset{\sim}{H}_{12} & \cdots & \underset{\sim}{G}_{1N} & \underset{\sim}{H}_{1N} \\ \underset{\sim}{K}_{11} & \underset{\sim}{L}_{11} & \underset{\sim}{K}_{12} & \underset{\sim}{L}_{12} & & \underset{\sim}{K}_{1N} & \underset{\sim}{L}_{1N} \\ \hline \underset{\sim}{G}_{21} & \underset{\sim}{H}_{21} & \underset{\sim}{G}_{22} & \underset{\sim}{H}_{22} & \cdots & \underset{\sim}{G}_{2N} & \underset{\sim}{H}_{2N} \\ \underset{\sim}{K}_{21} & \underset{\sim}{L}_{21} & \underset{\sim}{K}_{22} & \underset{\sim}{L}_{22} & & \underset{\sim}{K}_{2N} & \underset{\sim}{L}_{2N} \\ \hline \cdots & & \cdots & & & \cdots & \\ \hline \underset{\sim}{G}_{N1} & \underset{\sim}{H}_{N1} & \underset{\sim}{G}_{N2} & \underset{\sim}{H}_{N2} & \cdots & \underset{\sim}{G}_{NN} & \underset{\sim}{H}_{NN} \\ \underset{\sim}{K}_{N1} & \underset{\sim}{L}_{N1} & \underset{\sim}{K}_{N2} & \underset{\sim}{L}_{N2} & \cdots & \underset{\sim}{K}_{NN} & \underset{\sim}{L}_{NN} \end{array}\right] \qquad (5)$$

2.4. Transferfunction and aliasing expressions derived from the z-domain transfer matrix.

The frequency domain transfer function $\underset{\sim}{H}^{(ij)}(\omega)$ is the entry (ij) of the frequency domain transfer matrix $\underset{\sim}{H}(\omega)$ which relates the branch voltage vector $\underset{\sim}{U}(\omega)$ and the node voltage vector $\underset{\sim}{V}(\omega)$ and can be computed from :

$$\underset{\sim}{H}(\omega) = \sum_{k=1}^{N} \nu_k(\omega)\, e^{-j\omega t_{k+1}} \left(\sum_{\ell=1}^{N} e^{j\omega t_{\ell+1}}\, \underset{\sim}{H}_{k\ell}(e^{j\omega T}) \right) + \left[\frac{t_{k+1}-t_k}{T} - \nu_k(\omega) \right] \underset{\sim}{H}_{kk}(\infty), \tag{8}$$

where

$$\nu_k(\rho) \triangleq 2\{\sin\ \rho(t_{k+1}-t_k)/2 \quad \exp\ [j\rho(t_{k+1}-t_k)/2]\}\ /\ T\rho \tag{9}$$

and the matrices $\underset{\sim}{H}_{k\ell}(z)$ are given in (5).

Moreover the aliasing matrix is given by :

$$\underset{\sim}{X}(\omega,\omega+n\,\omega_s) = \sum_{k=1}^{N} \nu_k(\omega)\, e^{-j\omega\, t_{k+1}} \sum_{\ell=1}^{N} e^{j(\omega+n\,\omega_s)t_{\ell+1}}\, \underset{\sim}{H}_{k\ell}\,(e^{j(\omega+n\omega_s)T}) + [\nu_k(-n\,\omega_s)-\nu_k(\omega)]\ \exp[-n\,\omega_s t_{k+1}]\ \underset{\sim}{H}_{kk}(\infty) \tag{10}$$

2.5. The adjoint SC network.

Often, as in sensitivity and noise analysis it is useful to analyse the adjoint SC network $\tilde{\mathcal{N}}$ (1° Vandewalle, 1981) instead of the original network $\mathcal{N}$ because the number of computations can be greatly reduced. E,g. in the sensitivity analysis of p components (p+1) direct network analyses are needed whereas in the case of the use of the adjoint network only two analyses are needed : one for the direct and one for the adjoint network.

Construction of the adjoint SC network $\tilde{\mathcal{N}}$ of an SC network $\mathcal{N}$:

(1) Construct a duplicate of $\mathcal{N}$ and denote the sources by $\tilde{u}$ and $\tilde{w}$

(2) Modify the non-reciprocal elements (controlled sources) as follows. Interchange the controlling and controlled ports. Replace a VCVS by a QCQS and vice versa. Multiply the controlling coefficient by -1 in the case of a QCQS or VCVS.

(3) Reverse the time dependency of all time-varying elements in one period, i.e. the new clock signal $\tilde{\phi}_{rk}$ and the switching times $\tilde{t}i$ satisfy

$$\tilde{\phi}_{rk} = \phi_{r\tilde{k}}, \qquad \tilde{k} = N - k + 1$$

$$\tilde{t}_{\ell N+k} = t_{\ell N+1} + t_{(\ell+1)N+1} - t_{(\ell+1)N+2-k}$$

for $\ell = 0,1,2 \ldots$ and $k = 1,2, \ldots N$.

For example, by applying this algorithm on the SC filter section of figure 2(a) with clock signals (b) one obtains the adjoint SC network of figure 2(c) with signals (d).

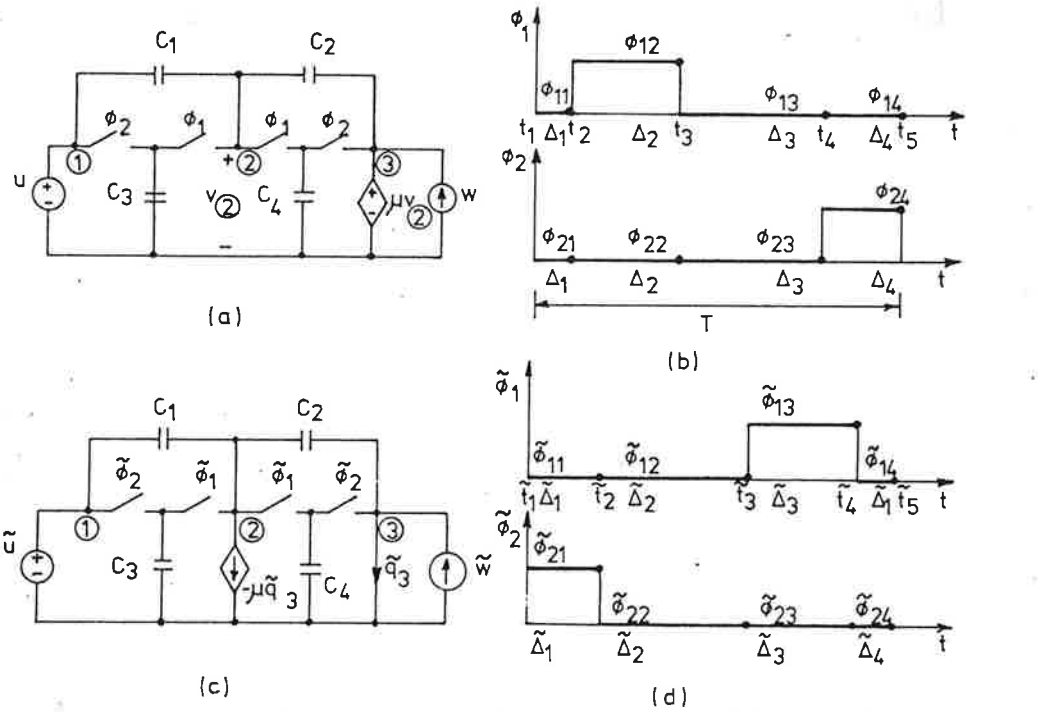


Figure 2. The SC network of (a) with clock signals (b) has an adjoint SC network (c) with clock signals (d)

The useful property of the adjoint switched capacitor network is given by theorem 4 in (1° Vandewalle, 1981).

Theorem :

Given a switched capacitor network $\mathcal{N}$ with z-domain transfer matrix $\underset{\sim}{M}$ and its adjoint network $\tilde{\mathcal{N}}$ with z-domain transfer matrix $\underset{\sim}{\tilde{M}}$, then

$$\underset{\sim}{\tilde{M}} = \underset{\sim}{M}^{\S T} \qquad (11)$$

Here § denotes a block row and column reversion defined as :

$$\left[\begin{array}{cccc|cccc} G_{11} & G_{12} & \dots & G_{1N} & H_{11} & H_{12} & \dots & H_{1N} \\ G_{21} & G_{22} & \dots & G_{2N} & H_{21} & H_{22} & \dots & H_{2N} \\ \dots & \dots & & \dots & \dots & \dots & & \dots \\ G_{N1} & G_{N2} & & G_{NN} & H_{N1} & H_{N2} & \dots & H_{NN} \\ \hline K_{11} & K_{12} & \dots & K_{1N} & L_{11} & L_{12} & \dots & L_{1N} \\ K_{21} & K_{22} & \dots & K_{2N} & L_{21} & L_{22} & \dots & L_{2N} \\ \dots & \dots & & \dots & \dots & \dots & & \dots \\ K_{N1} & K_{N2} & \dots & K_{NN} & L_{N1} & L_{N2} & \dots & L_{NN} \end{array}\right]^{\S} = \left[\begin{array}{cccc|cccc} G_{NN} & \dots & G_{N2} & G_{N1} & H_{NN} & \dots & H_{N2} & H_{N1} \\ \dots & & \dots & \dots & \dots & & \dots & \dots \\ G_{2N} & \dots & G_{22} & G_{21} & H_{2N} & \dots & H_{22} & H_{21} \\ G_{1N} & \dots & G_{12} & G_{11} & H_{1N} & \dots & H_{12} & H_{11} \\ \hline K_{NN} & \dots & K_{N2} & K_{N1} & L_{NN} & \dots & L_{N2} & L_{N1} \\ \dots & & \dots & \dots & \dots & & \dots & \dots \\ K_{2N} & \dots & K_{22} & K_{21} & L_{2N} & \dots & L_{22} & L_{21} \\ K_{1N} & \dots & K_{12} & K_{11} & L_{1N} & \dots & L_{12} & L_{11} \end{array}\right] \qquad (12)$$

For the computation of the adjoint equations, it is important to observe that the LU-factors of the adjoint $\tilde{\mathcal{N}}$ are equal to the LU factors of $\mathcal{N}$ after a transposition and a block reversion of the rows of the first and the columns of the second.

In the analysis of the direct network one column of $\underset{\sim}{M}$ can be calculated. In the analysis of the adjoint network one row of $\underset{\sim}{M}$ can be calculated. The proof of (11) is based on the fact that the matrices $\underset{\sim}{A}_k$, $\underset{\sim}{B}_k$, $\underset{\sim}{C}_k$, $\underset{\sim}{D}_k$ and $\underset{\sim}{E}_k$ of the description (4) of $\mathcal{N}$ and the matrices $\underset{\sim}{\tilde{A}}_k$, $\underset{\sim}{\tilde{B}}_k$, $\underset{\sim}{\tilde{C}}_k$, $\underset{\sim}{\tilde{D}}_k$ and $\underset{\sim}{\tilde{E}}_k$ of the corresponding description of $\tilde{\mathcal{N}}$ are related by

$$\begin{aligned} &\underset{\sim}{\tilde{A}}_k = \underset{\sim}{A}^T_{\tilde{k}} \quad , \quad \underset{\sim}{\tilde{D}}_k = \underset{\sim}{D}^T_{\tilde{k}} \\ &\underset{\sim}{\tilde{B}}_k = \underset{\sim}{C}^T_{\tilde{k}} \quad , \quad \underset{\sim}{\tilde{C}}_k = \underset{\sim}{B}^T_{\tilde{k}} \\ &\underset{\sim}{\tilde{E}}_k = \underset{\sim}{E}^T_{\tilde{k}+s}, \quad \text{where } s=1 \text{ if } k>1 \text{ and } s=1-N \text{ if } k=1 \text{ for } k = 1,2, \dots N \end{aligned} \qquad \text{with } \tilde{k} = N-k+1 \qquad (13)$$

(11) implies that the submatrices of the z-domain transfer matrix $\underset{\sim}{M}$ of $\mathcal{N}$ and $\underset{\sim}{\tilde{M}}$ of $\tilde{\mathcal{N}}$ are related by

$$\begin{aligned} &\underset{\sim}{G}_{k\ell}(z) = \underset{\sim}{\tilde{G}}^T_{\tilde{\ell}\tilde{k}}(z) \quad , \quad \underset{\sim}{H}_{k\ell}(z) = \underset{\sim}{\tilde{K}}^T_{\tilde{\ell}\tilde{k}}(z) \quad , \\ &\underset{\sim}{K}_{k\ell}(z) = \underset{\sim}{\tilde{H}}^T_{\tilde{\ell}\tilde{k}}(z) \quad , \quad \underset{\sim}{L}_{k\ell}(z) = \underset{\sim}{\tilde{L}}^T_{\tilde{\ell}\tilde{k}}(z) \quad , \end{aligned} \qquad (14)$$

where $\tilde{\ell} = N-\ell+1$, $\tilde{k} = N-k+1$ and for ℓ, k = 1, 2, ... N.

2.6. Sensitivity analysis (1° Vandewalle, 1981).

The sensitivity of $H^{(ij)}(\omega)$ on a parameter μ is given by

$$S_{\mu}^{H^{(ij)}}(\omega) = \frac{\mu}{H^{(ij)}(\omega)} \left\{ \sum_{k=1}^{N} \nu_k(\omega)\, e^{-j\omega t_{k+1}} \left(\sum_{\ell=1}^{N} e^{j\omega t_{\ell+1}} \frac{\partial H_{k\ell}^{(ij)}(e^{j\omega T})}{\partial \mu} \right) \right.$$

$$\left. + [(t_{k+1}-t_k)/T - \nu_k(\omega)] \frac{\partial H_{kk}^{(ij)}(\infty)}{\partial \mu} \right\} \qquad (15)$$

where in the case that μ is a capacitor C connecting node m to node n

$$\frac{\partial H_{k\ell}^{(ij)}(z)}{\partial C} = - \sum_{r=1}^{N} (\tilde{G}_{\tilde{r}\tilde{k}}^{(mi)} - \tilde{G}_{\tilde{r}\tilde{k}}^{(ni)}) (H_{r\ell}^{(mj)} - H_{r\ell}^{(nj)})$$

$$+ \sum_{r=1}^{N-1} (\tilde{G}_{\tilde{r}-1\ \tilde{k}}^{(mi)} - \tilde{G}_{\tilde{r}-1\ \tilde{k}}^{(ni)}) (H_{r\ell}^{(mj)} - H_{r\ell}^{(nj)}) \qquad (16)$$

$$+ z^{-1} (\tilde{G}_{1\ \tilde{k}}^{(mi)} - \tilde{G}_{1\ \tilde{k}}^{(ni)}) (H_{N\ell}^{(mj)} - H_{N\ell}^{(nj)})$$

with $\tilde{r} = N-r+1$ and $\tilde{k} = N-k+1$.

Or in the case that μ is a voltage controlled voltage source where the voltage in branch q is A times the voltage of node m with respect to node n

$$\frac{\partial H_{k\ell}^{(ij)}(z)}{\partial A} = \sum_{r=1}^{N} \tilde{K}_{\tilde{r}\tilde{k}}^{(qi)} (H_{r\ell}^{(mj)} - H_{r\ell}^{(nj)}) \qquad (17)$$

Or in the case that all capacitors are a known function of a parameter $\mu = \lambda$.

$$\frac{\partial H_{k\ell}^{(ij)}(z)}{\partial \lambda} = - \sum_{r=1}^{N} \underset{\sim}{\tilde{G}}_{\tilde{r}\tilde{k}}^{(.i)T} \frac{\partial \underset{\sim}{A}_{r}}{\partial \lambda} \underset{\sim}{H}_{r\ell}^{(.j)} + \sum_{r=1}^{N-1} \underset{\sim}{\tilde{G}}_{\tilde{r}-1\ \tilde{k}}^{(.i)T} \frac{\partial \underset{\sim}{A}_{r+1}}{\partial \lambda} \underset{\sim}{H}_{r\ell}^{(.j)}$$

$$+ z^{-1} \underset{\sim}{\tilde{G}}_{1\ k}^{(.i)T} \frac{\partial \underset{\sim}{A}_{1}}{\partial \lambda} \underset{\sim}{H}_{N\ell}^{(.j)} \qquad (18)$$

where $\underset{\sim}{H}_{r\ell}^{(.j)}$ is again the j-th column of $\underset{\sim}{H}_{r\ell}$, and $\frac{\partial \underset{\sim}{A}_k}{\partial \lambda}$ is known from the design.

Other sensitivities and the group delay are given in (1• Vandewalle, 1981).

3. Efficient matrix setup.

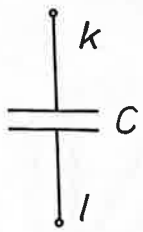
In this section the setup of the MNA-matrix by use of "stamps" is described. Afterwards it is shown how composite switch branches can be used for an efficient matrix formulation.

3.1. Matrix setup by use of stamps.

The setup of the z-domain MNA matrix can be automated by an accumulation of "stamps" (Ho, 1975) for each network element in the matrix. These stamps represent the contribution of a certain element in the equations. Due to the coupling between the consecutive timeslots the stamps are extended for the z-domain MNA-matrix.

Table 1 contains a list of some frequently used elements in SC-analysis as implemented in DIANA.

Capacitor C



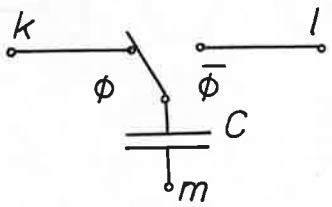
for phases i≠1 :

	k_{i-1}	l_{i-1}	k_i	l_i
k_i	-C	C	C	-C
l_i	C	-C	-C	C

for phase i = 1 :

	k_1	l_1	k_N	l_N
k_1	C	-C	$-C\,z^{-1}$	$C\,z^{-1}$
l_1	-C	C	$C\,z^{-1}$	$-C\,z^{-1}$

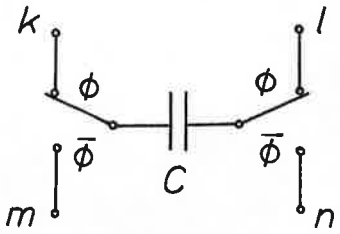
S3-switch



for phases i≠1*

	k_{i-1}	l_{i-1}	m_{i-1}	k_i	l_i	m_i
k_i	$-\phi_i\phi_{i-1}C$	$-\phi_i\bar{\phi}_{i-1}C$	$\phi_i C$	$\phi_i C$	0	$-\phi_i C$
l_i	$-\bar{\phi}_i\phi_{i-1}C$	$-\bar{\phi}_i\bar{\phi}_{i-1}C$	$\bar{\phi}_i C$	0	$\bar{\phi}_i C$	$-\bar{\phi}_i C$
m_i	$\phi_{i-1}C$	$\bar{\phi}_{i-1}C$	-C	$-\phi_i C$	$-\bar{\phi}_i C$	C

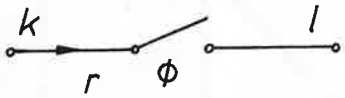
Four node switch



for phases i≠1*

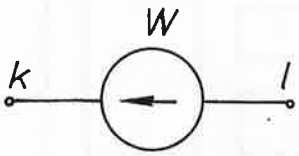
	k_{i-1}	l_{i-1}	m_{i-1}	n_{i-1}	k_i	l_i	m_i	n_i
k_i	$-\phi_i\phi_{i-1}C$	$\phi_i\phi_{i-1}C$	$-\phi_i\bar{\phi}_{i-1}C$	$\phi_i\bar{\phi}_{i-1}C$	$\phi_i C$	$-\phi_i C$	0	0
l_i	$\phi_i\phi_{i-1}C$	$-\phi_i\phi_{i-1}C$	$\phi_i\bar{\phi}_{i-1}C$	$-\phi_i\bar{\phi}_{i-1}C$	$-\phi_i C$	$\phi_i C$	0	0
m_i	$-\bar{\phi}_i\phi_{i-1}C$	$\bar{\phi}_i\phi_{i-1}C$	$-\bar{\phi}_i\bar{\phi}_{i-1}C$	$\bar{\phi}_i\bar{\phi}_{i-1}C$	0	0	$\bar{\phi}_i C$	$-\bar{\phi}_i C$
n_i	$\bar{\phi}_i\phi_{i-1}C$	$-\bar{\phi}_i\phi_{i-1}C$	$\bar{\phi}_i\bar{\phi}_{i-1}C$	$-\bar{\phi}_i\bar{\phi}_{i-1}C$	0	0	$-\bar{\phi}_i C$	$\bar{\phi}_i C$

S2-switch



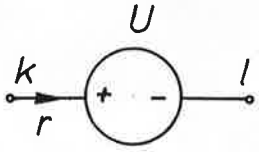
	k_i	l_i	r_i
k_i	0	0	ϕ_i
l_i	0	0	$-\phi_i$
r_i	ϕ_i	$-\phi_i$	$\bar{\phi}_i$

charge source



	k_i	l_i	source vector
k_i	0	0	W
l_i	0	0	-W

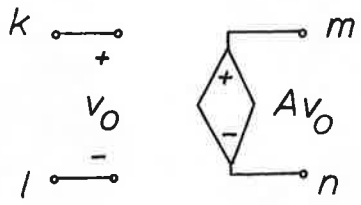
Voltage source



	k_i	l_i	r_i	source vector
k_i	0	0	1	0
l_i	0	0	-1	0
r_i	1	-1	0	u

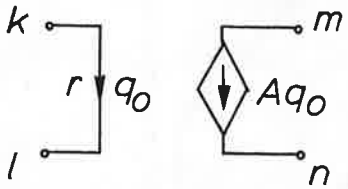
* Only the contribution of the stamp to the equations for phases i=2...N is given. Of course for phase i=1 a modification with z^{-1} as in the second stamp of the capacitor has to be used.

voltage controlled voltage source



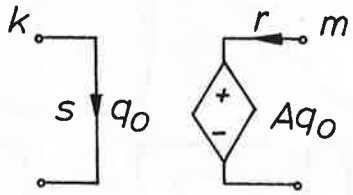
	k_i	l_i	m_i	n_i	r_i
k_i	0	0	0	0	0
l_i	0	0	0	0	0
m_i	0	0	0	0	1
n_i	0	0	0	0	-1
r_i	-1	A	1	-1	0

charge controlled charge source



	k_i	l_i	m_i	n_i	r_i
k_i	0	0	0	0	1
l_i	0	0	0	0	-1
m_i	0	0	0	0	A
n_i	0	0	0	0	-A
r_i	1	-1	0	0	0

charge controlled voltage source



	k_i	l_i	m_i	n_i	r_i	s_i
k_i	0	0	0	0	0	1
l_i	0	0	0	0	0	-1
m_i	0	0	0	0	1	0
n_i	0	0	0	0	-1	0
r_i	0	0	1	-1	0	-A
s_i	1	-1	0	0	0	0

Table 1. Stamps for phase i in the z-domain MNA-matrix of ideal elements used in SC-analysis.

3.2. The use of composite switch branches.

As can be noticed from table 1, composite switch branches (S3 and S4) can also be used. Often this is also called a macromodel.
Their use allows an initial compaction of the network equations. When using the S3-switch model compared to two S2 switches and one capacitor, three internal variables and equations can be saved in each phase. By use of an S4-switch model instead of four switches and one capacitor, six variables and equations can be saved in each phase. This of course implies that one (resp. two) node voltages and charges through switches are unknown.

For the example of fig. 1. a S3-switch model can be used instead of S1, S2 and C1. This results in the following z-domain MNA-matrix.:

$$
\begin{array}{c}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 1' \\ 2' \\ 3' \end{array}
\left[\begin{array}{c|c|c|c|c|c}
1 & 2 & 3 & 1' & 2' & 3' \\
\hline
0 & 0 & 1 & 0 & 0 & 0 \\
0 & C_2 & 0 & 0 & -C_2 z^{-1} & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & C_1 & -C_1 & 1 \\
0 & -C_2 & 0 & -C_1 & C_1+C_2 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
\cdot
\left[\begin{array}{c} V_{11} \\ V_{31} \\ \hline Q_{11} \\ \hline V_{12} \\ V_{32} \\ \hline Q_{12} \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ \hline V_{i1} \\ \hline 0 \\ 0 \\ \hline V_{i2} \end{array}\right]
\end{array}
\qquad (19)
$$

Equations (7) and (19) are equivalent in the sense that they represent the matrix which computes the same output variables that are of interest to us.

By using this composite switch branch model six variables and six equations are implicitely eliminated in the original MNA-matrix (which was already small compared to the tableau matrix). For this simple example the gain in computational effort is very big by using composite switch branches.

But even with the inclusion of these composite switch branches still relatively large matrices are obtained when the circuit size increases, and this especially so when several timeslots are used. For the two phase fifth order elliptic lowpass filter (2• Jacobs, 1978) shown in fig. 4. one obtains with the use of S2 switches a z-domain MNA-matrix with dimension $N = 112$. If use is made of S3 and S4 composite branch switches this matrix dimension is reduced to $N = 34$.

The dimension of the matrix is very important because it determines the matrix solution time. If appropriate sparse matrix techniques are used, the solution time of the z-domain MNA-matrix of a "normal" SC network is approximately linear to the dimension of this matrix.
Therefore it is important that one has a smaller matrix if a computation over several frequency values is desired.

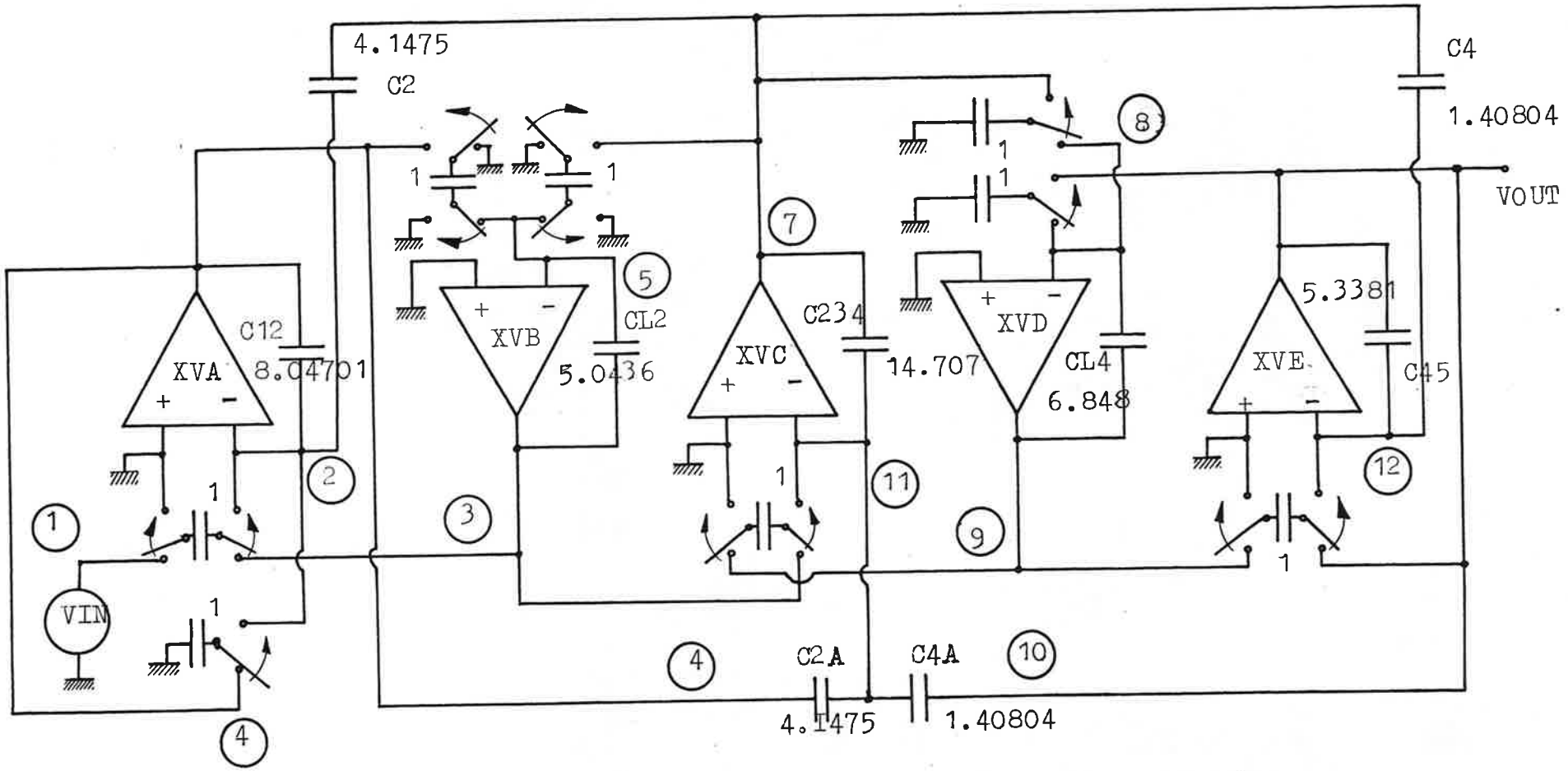

4.1475
C2
C4
1.40804
VOUT
C12
8.04701
XVA
XVB
CL2
5.0436
XVC
C234
14.707
XVD
CL4
6.848
XVE
5.3381
C45
VIN
C2A
4.1475
C4A
1.40804

fig. 4.

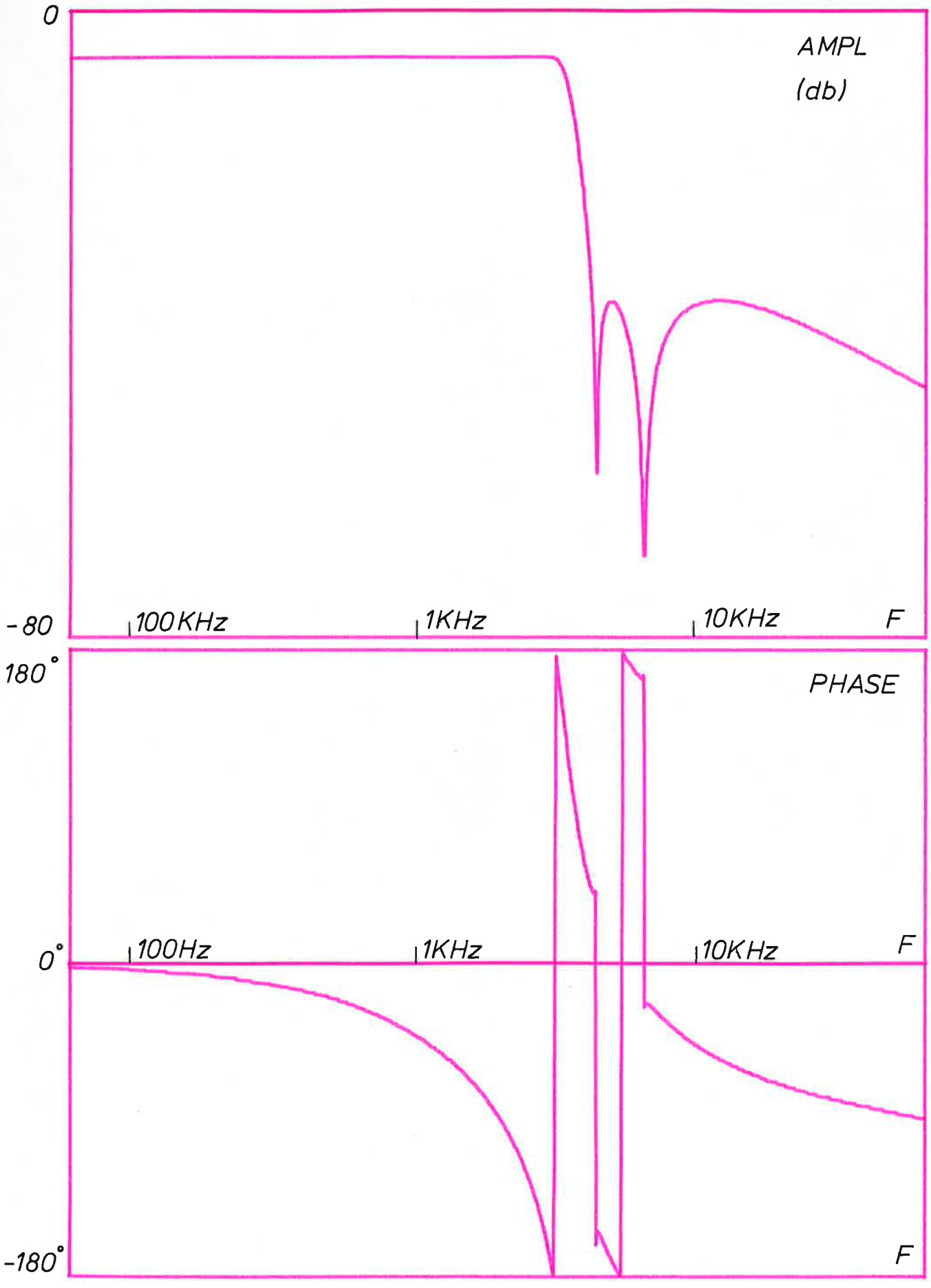

0
AMPL
(db)
-80
100KHz
1KHz
10KHz
F
180°
PHASE
0°
100Hz
1KHz
10KHz
F
-180°
F

fig. 5

```
ESAT ELLIPTICAL FILTER
.SCFREQ NLIN=60 FSTART=0 FSTOP=16KHZ FSAMPLE=128KHZ
PRINT 10
;CLOCKING
INPUT CL V-T=0 0 2 1 0 2 CYC
;CIRCUIT
VIN 1 0 1
C12 4 2 8.4701
C234 7 11 14.707
CL2 5 3 5.0436
CL4 8 9 6.848
C45 10 12 5.3381
C4 12 7 1.40804
C2 7 2 4.1475
C2A 11 4 4.1475
C4A 11 10 1.40804
XVA 4 0 0 2 1K
XVB 3 0 0 5 1K
XVC 7 0 0 11 1K
XVD 9 0 0 8 1K
XVE 10 0 0 12 1K
S31 0 4 2 CL 0 C=1 VT=1
S42 1 3 0 2 CL 0 C=1 VT=1
S43 0 5 4 0 CL 0 C=1 VT=1
S44 0 5 7 0 CL 0 C=1 VT=1
S45 9 3 0 11 CL 0 C=1 VT=1
S36 0 8 7 CL 0 C=1 VT=1
S37 0 8 10 CL 0 C=1 VT=1
S49 9 10 0 12 CL 0 C=1 VT=1
.SENS CAP=C12 C2  CL2
.END
```

fig. 6

4. Efficient compaction.

In this section a matrix compaction algorithm is presented by which the equations (4) can be compacted into a small set of equations. This minimal set of equations can than be used in an efficient solution process as described hereafter. The size of this equations set is usually considerably smaller than that obtained with other approaches (Brglez, 1980), (Kuo, 1979), (Tsividis, 1979).

4.1. Compaction of the equations (4).

Observe that the matrix (4) is a set of linear equations which can be written as :

$$[\underset{\sim}{A} + z^{-1}\ \underset{\sim}{B}]\ \underset{\sim}{\xi} = \underset{\sim}{\zeta} \qquad (20)$$

These equations can be obtained by accumulating the contributions of the different stamps of table 1 into the z-domain MNA-matrix.

We assume that the SC circuit obeys the following topological conditions. In the case that the independent voltage and charge sources are the only active components, then in each timeslot there may not be any cutset of charge sources and open switches and no loop of voltage sources and closed switches. In this case equation (20) can be solved. Moreover for almost all the practical circuits which contain dependent sources the matrix $\underset{\sim}{A}$ will be nonsingular.

By compaction we mean the Gauss elimination of all internal variables in (20) (matrix columns !) which are of no further interest for the following calculations by using all the equations in (20) (matrix rows !) which are of no further interest for the following calculations so that we obtain a compacted set of equations :

$$[\underset{\sim}{A}' + z^{-1}\ \underset{\sim}{B}']\ \underset{\sim}{\xi}' = \underset{\sim}{\zeta}' \qquad (20.a)$$

The variables which are of further interest are the desired output variables of the simulation and certain variables necessary for the sensitivity analysis. Equations which are of further interest are input variables and equations corresponding to desired output variables of the adjoint network in sensitivity analysis.

Notice the duality between rows and columns in the elimination proces because of the use of the adjoint switched capacitor network.

Example : Suppose one would like to analyse the transferfunction from node 1 to node 3 in the filter of fig. 1. Then V_{31} and V_{32} are the desired outputs while equation 4 and equation 4' are equations of further interest. We have marked these rows and columns in fig. 1. by a # -sign.

It should be mentioned that our solution process can also be applied in other approaches (Brglez, 1980), (Kuo, 1979), (3° Tsividis, 1979), (Hokenek, 1980) to setup the equations.

The analysis below could even be adapted to the timedomain analysis of ideal (no transient effects !) SC circuits, by a back transformation of the z-domain equations (20.a) to the time domain where z^{-1}-operator corresponds to a delay in time :

$$\underset{\sim}{A}' \underset{\sim}{\xi}'(nT) = -\underset{\sim}{B}' \underset{\sim}{\xi}'(n-1)T) + \underset{\sim}{\zeta}'(nT) \qquad (20.b)$$

Variables in different phases of the solutionvector $\underset{\sim}{\xi}'$ must of course be put back in the right timesequence.

Observe that the columns contained in the n-1 first block columns of matrix (4) do not depend on z^{-1}. This allows to perform Gauss eliminations which are valid for all z^{-1}. Also LU decomposition may be performed when taking pivot elements in parts which do not depend on z^{-1} because LU-decomposition is essentially the same as Gauss elimination.

In the last block column the columns corresponding to the charge variables are also z^{-1}-independent. The rows in the last block column corresponding to the branch constitutive relations are also z^{-1}-independent. These ideas are carried out more formally in appendix 1.

4.2. Compaction process*.

* The compaction process described below is very general in the sense that it can be applied to all matrix equations of the form : $[\underset{\sim}{A} + \lambda \underset{\sim}{B}] . \underset{\sim}{X} = \underset{\sim}{Y}$ where λ is a complex parameter.

Suppose the following z-domain matrix :

$$\begin{bmatrix} \underset{\sim}{F}_1 & & & & -\underset{\sim}{G}_1 z^{-1} \\ -\underset{\sim}{G}_2 & \underset{\sim}{F}_2 & & & \\ & -\underset{\sim}{G}_3 & \underset{\sim}{F}_3 & & \\ & & \ddots & & \\ & & & \underset{\sim}{F}_{N-1} & \\ & & & -\underset{\sim}{G}_N & \underset{\sim}{F}_N \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{X}_1 \\ \underset{\sim}{X}_2 \\ \underset{\sim}{X}_3 \\ \vdots \\ \underset{\sim}{X}_{N-1} \\ \underset{\sim}{X}_N \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}_1 \\ \underset{\sim}{Y}_2 \\ \underset{\sim}{Y}_3 \\ \vdots \\ \underset{\sim}{Y}_{N-1} \\ \underset{\sim}{Y}_N \end{bmatrix} \qquad (22)$$

$$\underset{\sim}{M}^{-1}$$

where $\underset{\sim}{F}_i$ is a main submatrix of phase i. For the MNA formulation this results in :

$$\underset{\sim}{F}_i = \begin{bmatrix} \underset{\sim}{A}_i & \underset{\sim}{B}_i \\ \underset{\sim}{C}_i & \underset{\sim}{D}_i \end{bmatrix}$$

$\underset{\sim}{G}_i$ is a coupling submatrix of phase i which for a MNA matrix looks like :

$$G_i = \begin{bmatrix} -\underset{\sim}{E}_i & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}$$

$\underset{\sim}{X}_i$ is the unknown vector of phase i :

$$\underset{\sim}{X}_i = \begin{bmatrix} \underset{\sim}{V}_i \\ \underset{\sim}{Q}_i \end{bmatrix}$$

and $\underset{\sim}{S}_i$ is the source vector of phase i.

$$\underset{\sim}{Y}_i = \begin{bmatrix} \underset{\sim}{W}_i \\ \underset{\sim}{U}_i \end{bmatrix}$$

Sparse matrix techniques, especially tailored for our class of problems are derived. The advantage of this approach is that much less core-memory is required to store the matrix and that the computationtimes increase linearly with the circuit complexity compared to a full matrix implementation.

Part 1. Compaction of the first N-1 submatrices (phases).

Depending upon the available computer memory the compaction of the equations of the first N-1 phases can be done one at a time (gradual) or all at once. If compaction is performed gradually less memory is necessary but more sparse matrix fillins are generated in the reordering process. If compaction of all first N-1 timeslots is done at once less fillins occur but the ordering and pivotting algorithms will take more time. The time necessary for matrix reordering and pivotting increases with the thirth power of the submatrix

dimension to be ordered. Therefore for circuits with many timeslots it is better to use the gradual compaction proces. In the DIANA-program both kinds of analysis can be specified (default : gradual, can be overridden by the-GRADUAL-option).

Step 1.

1.1. Setup and enter $\underset{\sim}{G}_1$ in the sparse matrix system.

Step 2.

2.1. Setup and enter $\underset{\sim}{F}_i$ and $\underset{\sim}{G}_{i+1}$ in the sparse matrix system.

2.2. If i=N-1 also setup and enter $\underset{\sim}{F}_N$ in the sparse matrix system.
This will allow to determine a minimum number of fillins during the reordering process.

2.3. Singletonalgorithm : (Ho, 1975) Reorder singletons (rows and columns with only one element in them) in $\underset{\sim}{F}_i$ in the first part of the matrix $\underset{\sim}{F}_i$. (These singletons can be assigned as initial pivots for either non blocked rows and columns or blocked rows or columns. The singletons originate either from grounded voltage sources and grounded closed switches or from current sources and open switches).

2.4. Perform a pivoting algorithm in order to obtain a strong matrixdiagonal which is different from zero.
This means that the diagonal elementvalues may not be small compared to the other matrix element values. This guarantees the numerical stability and accuracy of the solution process.

2.5. Block the rows in $\underset{\sim}{F}_i$ corresponding to input sources and desired outputs of the adjoint network $\tilde{\mathcal{N}}$ in phase i.

2.6. Block the columns in $\underset{\sim}{F}_i$ corresponding to desired outputs and input sources of the adjoint network in phase i.

2.7. Delete rows and columns corresponding to singletons of nonblocked rows and non blocked columns.

2.8. Make pivots free which correspond to rows and columns which are not blocked and order them in the first (freed) part of the matrix $\underset{\sim}{F}_i$.
A pivot is said to be free if its corresponding row and column are both not blocked and if they are made available for Gauss elimination.

2.9. Reorder the pivots of the freed part in order to cause a minimum number of fillins in the sparse matrix system.

2.10. Perform a Gauss-elimination of the whole matrix part that has last been freed. Hereafter entries of the sparse matrix system corresponding to eliminated rows and columns can be deleted in order to obtain an efficient memory use and fast execution time.

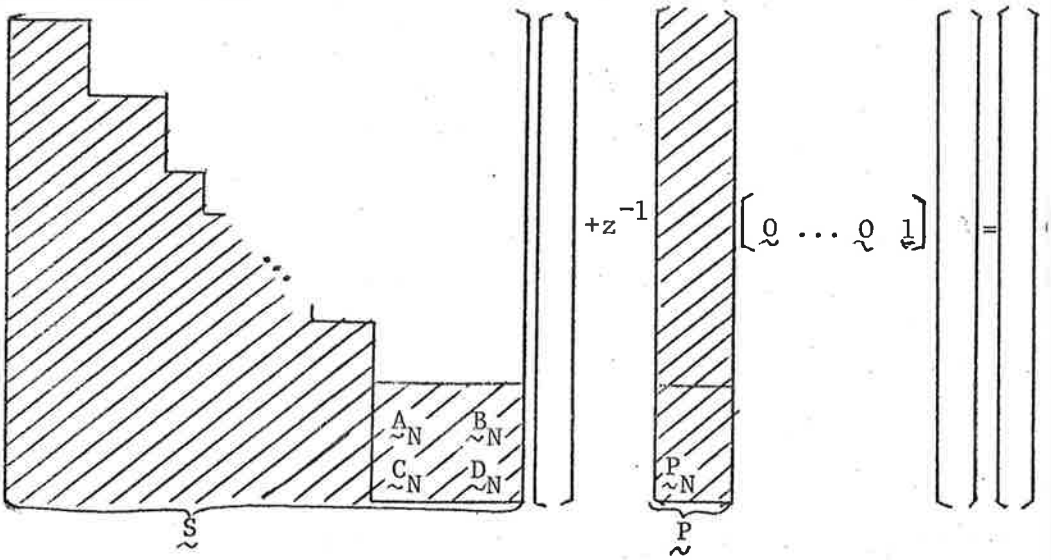
2.11. Examine if non-blocked rows or columns of the non-free part can be freed by performing a column interchange.
If so, free the non free part.
The new pivots are chosen so that their absolute value is greater than a certain threshold value V_T in order to maintain a strong diagonal.

2.12. If there is at least one pivot freed in step 2.11. repeat from 2.9.

2.13. Reorder the remaining non-freed part of $\underset{\sim}{F}_i$ for a minimum number of fillins, and perform a partial LU decomposition in this part.

2.14. If $i < N-1$ repeat from 2.1.

At this point in the compaction process the matrix equation (4) is now reduced to :



(23)

where only nonzero real elements occur in the shaded areas of $\underset{\sim}{S}$ and $\underset{\sim}{P}$.

Part 2. Compaction of the last timeslot.

It is our intention to compact the phase N equations so that the compacted equation is still of the form (20.a). More generally it is possible but not advisable to compact until transferfunctions are obtained if one works with formula manipulation in the field of rational functions of z.

Step 3.

3.1. Detect singletons and order them in the first part of $\underset{\sim}{F}_N$.

3.2. Pivoting in the last phase part of S in order to obtain a nonzero diagonal.

3.3. Block rows in $\underset{\sim}{F}_N$ corresponding to input sources and desired outputs of the adjoint network in the last phase.

3.4. Block columns in $\underset{\sim}{F}_N$ corresponding to input sources of the adjoint network and desired outputs of the direct network in the last phase.

3.5. Delete rows and columns corresponding to singletons of non-blocked rows and non-blocked columns.

3.6. Rowalgorithm :

3.6.1. Make pivots free which correspond to rows which are z^{-1}-independent and not blocked and which correspond to columns which are not blocked. See in the appendix 1 for the definition of z^{-1}-independent row and column. (Note that rows of $\underset{\sim}{C}_N$ and $\underset{\sim}{D}_N$ are z^{-1}-independent during the first pass of the row algorithm).

3.6.2. Reorder the last free pivots for a minimum number of fillins. Perform a Gauss elimination on the free part. Delete the matrixentries corresponding to the eliminated rows and columns.
This can be done all at once because (corollary 2 in appendix 1) z^{-1} independent rows remain z^{-1} independent during Gauss elemination.

3.6.3. Examine for non blocked rows if they are z^{-1} independent and see if a columninterchange of the pivotcolumn with a non blocked column can result in a new free pivot. Search all new pivots in this way. The new pivots are chosen in $\underset{\sim}{S}$ (23) so that a strong diagonal can be maintained.

3.6.4. If there is at least one pivot free in 3.6.3. then repeat from 2.9.

3.7. Column algorithm.

3.7.1. dual to 3.6.1.

3.7.2. dual to 3.6.2.

3.7.3. dual to 3.6.3.

3.7.4. dual to 3.6.4.

3.8. Repeat row- and column algorithms 3.6. and 3.7. until no additional pivots can be freed and eliminated anymore.

3.9. At this stage still a row and column algorithm could be performed but now also with blocked rows and blocked columns and by using LU decomposition instead of Gauss elimination and without deleting matrix entries. This would result in the fact that for each frequency value an LU-decomposition had to be done on a minimal dimension matrix. This step 3.9. is not implemented in DIANA because this would reduce the total computation time only by a few percent.

3.10. Reorder the remaining rows and columns of phase N so that a minimum number of "fillins" is created by LU decomposition of this part.

The result of this compaction process can be written as :

$$\begin{bmatrix} \underset{\sim}{L} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{S} & \underset{\sim}{I} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{R}\, z^{-1} \\ \underset{\sim}{0} & \underset{\sim}{H}+z^{-1}\underset{\sim}{P} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{U} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{X}' \\ \underset{\sim}{X}'_N \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}' \\ \underset{\sim}{Y}'_N \end{bmatrix} \qquad (24)$$

where $\underset{\sim}{L}$ is a lower triangular matrix and $\underset{\sim}{U}$ is an upper triangular matrix. Only the real matrices $\underset{\sim}{L}$, $\underset{\sim}{U}$, $\underset{\sim}{S}$, $\underset{\sim}{R}$, $\underset{\sim}{H}$ and $\underset{\sim}{P}$ need to be stored.

Example : If we apply the compaction process described above to the filter in fig. 1. with a MNA matrix (7) and the row and column constraints indicated above we obtain the following matrix equation after step 3.10.

$$\begin{array}{c} \\ 4 \\ 3 \\ 4' \\ 3' \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 1' & 3' \end{array} \\ \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 1' & 3' \end{array} \\ \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & -C_2\, z^{-1} \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 0 & -C_1 & C_1+C_2-C_2 z^{-1} \end{array}\right] \end{array} \begin{bmatrix} V_{11} \\ V_{31} \\ \hdashline V_{12} \\ V_{32} \end{bmatrix} \quad \begin{bmatrix} V_{i1} \\ 0 \\ \hdashline V_{i2} \\ 0 \end{bmatrix} \qquad (25)$$

This solution corresponds to (24) but in the form :

$$\begin{bmatrix} \underset{\sim}{L} & \underset{\sim}{0} \\ \underset{\sim}{S} & \underset{\sim}{I} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{U} & \underset{\sim}{R}\, z^{-1} \\ \underset{\sim}{0} & \underset{\sim}{H} + z^{-1} \underset{\sim}{P} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{X}' \\ \underset{\sim}{X}'_N \end{bmatrix} = \begin{bmatrix} \underset{\sim}{Y}' \\ \underset{\sim}{Y}'_N \end{bmatrix}$$

Example : The matrix of the fifth order filter of fig. 4. reduces to a 8 x 8 matrix after compaction if only the input-output transferfunction is to be calculated. This is the case as well for the use of S2, S3 - as for S4 composite switch branches.

It is our experience that the matrices can always be compacted to a dimension that is less than or equal to the number of desired in- and outputs in the desired phases plus the order of the filter.

4.3. Solution process for a certain ω_m-value.

Step 1.

For any ω_m , set $z=e^{j\omega_m T}$ and perform the LU decomposition of the last phase, which is useful for all excitations $\underset{\sim}{Y}'$, $\underset{\sim}{Y}'_N$. Since the matrix $\underset{\sim}{M}$ is nonsingular for almost all values of ω_m, this LU decomposition can be performed :

$$\underset{\sim}{H} + e^{-j\omega_m T}\, \underset{\sim}{P} = \underset{\sim}{L}_N\, \underset{\sim}{U}_N$$

Step 2.

For any excitation $\underset{\sim}{Y}'$, $\underset{\sim}{Y}'_N$ the response $\underset{\sim}{X}'$, $\underset{\sim}{X}'_n$ can now be computed as follows.

$$\begin{aligned} \underset{\sim}{Z} &= \underset{\sim}{L}^{-1}\, \underset{\sim}{Y} , \\ \underset{\sim}{F}_N &= \underset{\sim}{Y}_N - \underset{\sim}{S}\, \underset{\sim}{Z} , \\ \underset{\sim}{X}_N &= \underset{\sim}{U}_N^{-1}\, \underset{\sim}{L}_N^{-1}\, \underset{\sim}{F}_N , \\ \underset{\sim}{M} &= \underset{\sim}{Z} - e^{j\,\omega_m T}\, \underset{\sim}{R}\, \underset{\sim}{X}_N , \\ \underset{\sim}{X} &= \underset{\sim}{U}^{-1}\, \underset{\sim}{M} \end{aligned} \qquad (25)$$

The reader can verify for himself by simple substitutions in (19) that this is indeed a correct solution.

If one is interested in a solution of the adjoint equations, step 1 does noet have to be repeated.
If the contributions of the different phases to the vectors $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are organized in decreasing order, we have to solve $\underset{\sim}{M}^{-1\,T}\underset{\sim}{X} = \underset{\sim}{Y}$.
Using the matrices of (24) this equation can be decomposed as

$$\begin{bmatrix} \underset{\sim}{U}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{R}^T z^{-1} & \underset{\sim}{H}^T + z^{-1} \underset{\sim}{P}^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{S}^T \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{L}^T & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\tilde{X}} \\ \underset{\sim}{\tilde{X}}_1 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\tilde{Y}} \\ \underset{\sim}{\tilde{Y}}_1 \end{bmatrix} \qquad (26)$$

This allows to describe the different computations for the adjoint network.

Step $\tilde{4}$: Compute

$$\begin{aligned} \underset{\sim}{\tilde{Z}} &= \underset{\sim}{U}^{T-1}\, \underset{\sim}{\tilde{Y}} \\ \underset{\sim}{\tilde{F}}_1 &= \underset{\sim}{\tilde{Y}}_1 - e^{-j\,\omega_m T}\, \underset{\sim}{R}\, \underset{\sim}{\tilde{Z}} \\ \underset{\sim}{\tilde{X}}_1 &= \underset{\sim}{L}_N^{T-1}\, \underset{\sim}{U}_N^{T-1}\, \underset{\sim}{\tilde{F}}_1 \\ \underset{\sim}{\tilde{M}} &= \underset{\sim}{\tilde{Z}} - \underset{\sim}{S}^T\, \underset{\sim}{\tilde{X}}_1 \\ \underset{\sim}{\tilde{X}} &= \underset{\sim}{L}^{T-1}\, \underset{\sim}{\tilde{M}} \end{aligned} \qquad (27)$$

Careful inspection shows that the number of computations required by this algorithm increases linearly with the number N of phases. This is certainly important for circuits with many phases, but also for circuits with two phases the algorithm is very fast.
Observe that the matrix $\underset{\sim}{P}$ characterizes the essential signal circulation phenomena of a SC network.

4.4. Parametric analysis.

In a parametric analysis one wants to study the influence of large parametric changes of one or more network component(s) (e.g. a capacitorvalue or the amplification of a dependant source) on a transferfunction $H^{(ij)}$ or a sensitivity.

In this case the above compaction formalism can still be used. One only has to block the row(s) and column(s) corresponding to these parameter entries in the matrix. These parameterentries have to be ordered last in the matrix so that the LU decompositionproces can be delayed until the compaction of the rest of the matrix has been performed.

Afterwards for all parametervalues of interest to the user (the compaction doesn't have to be repeated) the matrix entries can be filled so that the decomposition can be performed on this last part. The ensuing solution process remains the same as in the usual analysis.

When the compaction process is further pursued so that transferfunction derivation and pole-zero extraction is possible one can do a root-locus analysis with the help of this parameteranalysis formalism. A complete compaction procedure followed by a pole zero extraction can have numerical problems due to ill conditioning and becomes easily quite involved for large circuits. From a numerical stability point of view it would be better to solve the "eigenvalue"-problem of the pencil (21a) e.g. by the methods proposed by (Van Dooren, 1979).

4.5. Non ideal effects.

In the analysis of the nonideal effects (inclusion of transient effects, resistors, opamp poles and zeros) (Rabaey, 1981) also a compaction can be performed till the order of the system plus the number of desired in- and outputs. This compaction can be efficient if higher order Padé approximations are used where several matrix manipulations are required. But one should remark that the order of a SC circuit with resistors in the switches will often be higher because there are more independent capacitors in it, so that the compaction process does not bring about such a reduction as in the ideal case.

5. Efficient organisation of frequency, aliasing and sensitivity computations (2° Vandewalle, 1981).

Only two sets of linear equations of the form (24) have to be solved at each ω_m of interest in order to obtain the frequency, aliasing and sensitivity characteristics at ω_m. This is considerably better than what is obtained for the time domain analysis. Our algorithm makes full advantage of the following observations. The frequency and aliasing characteristics require linear combinations of certain entries of the matrix $\underset{\sim}{M}$. These entries and those needed for the sensitivities can often be computed more efficiently from the matrix $\underset{\sim}{M}$ of the adjoint switched capacitor network. By choosing the excitation $\underset{\sim}{Y}$ appropriately certain linear combinations are automatically obtained in (24). The contribution from the continuous coupling can be obtained by setting $z = \infty$ in (24).

Step 1. Apply an excitation $\underset{\sim}{Y}$ where component j of $\underset{\sim}{U}_k$ is 1 and all other components of $\underset{\sim}{Y}$ are zero and solve (24) with $z=\infty$. The solution $\underset{\sim}{X}$ is then pertitioned into N vectors.

$$\begin{bmatrix} \underset{\sim}{\lambda}_k \\ \underset{\sim}{\sigma}_k \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{kk}^{(.j)}(\infty) \\ \underset{\sim}{L}_{kk}^{(.j)}(\infty) \end{bmatrix} \qquad \text{for } k = 1, 2, \ldots N$$

where $\underset{\sim}{S}^{(.j)}$ stands for the j-th column of the matrix $\underset{\sim}{S}$.

Apply an excitation $\underset{\sim}{Y}$ where component i of $\underset{\sim}{W}_k$, k=N-k+1 is 1 and all other components of $\underset{\sim}{Y}$ are zero and solve these equations of $z=\infty$. The solution $\underset{\sim}{X}$ is then partitioned into N vectors

$$\begin{bmatrix} \tilde{\underset{\sim}{\lambda}}_k \\ \tilde{\underset{\sim}{\sigma}}_k \end{bmatrix} = \begin{bmatrix} \underset{\sim}{G}_{kk}^{(i.)T}(\infty) \\ \underset{\sim}{K}_{kk}^{(i.)T}(\infty) \end{bmatrix} \qquad k = 1, 2, \ldots N$$

Step 2. Solve (24) at each $\omega_m \in \Omega$ for the original circuit under the following voltage excitation in branch j (not to be confused with the $j=\sqrt{-1}$).

$$\underset{\sim}{U}^{(j)} = e^{j\omega_m t_{\ell+1}} \quad \text{for } \ell = 1, 2, \ldots N$$

and all other components of Y zero. The solution X is then partitioned into N vectors with

$$\begin{bmatrix} \underset{\sim}{\Lambda}_k \\ \underset{\sim}{\Sigma}_k \end{bmatrix} = \sum_{\ell=1}^{N} e^{j\omega_m t_{\ell+1}} \begin{bmatrix} \underset{\sim}{H}_{k\ell}^{(.j)}(e^{j\omega_m T}) \\ \underset{\sim}{L}_{k\ell}^{(.j)}(e^{j\omega_m T}) \end{bmatrix} \qquad \text{for } k = 1, 2, \ldots N.$$

Solve the adjoint equations at ω_m for

$$\tilde{\underset{\sim}{W}}_k^{(i)} = \nu_k(\omega_m)\, e^{-j\omega_m t_{k+1}} \quad \text{for } k = 1, \ldots N$$

where

$$\nu_k(\ell) = 2\left\{\sin\left[\ell(t_{k+1}-t_k)/2\right] \exp\left[j\ell(t_{k+1}-t_k)/2\right]\right\}/T$$

and all other components of $\tilde{\underset{\sim}{Y}}$ zero. The solution is then partitioned into N vectors with

$$\begin{bmatrix} \tilde{\underset{\sim}{\Lambda}}_\ell \\ \tilde{\underset{\sim}{\Sigma}}_\ell \end{bmatrix} = \sum_{k=1}^{N} \nu_k(\omega_m)\, e^{-j\omega_m t_{k+1}} \begin{bmatrix} \underset{\sim}{G}_{k\ell}^{(i.)T}(e^{j\omega_m T}) \\ \underset{\sim}{K}_{k\ell}^{(i.)T}(e^{j\omega_m T}) \end{bmatrix} \qquad \text{for } \ell = 1,2, \ldots N.$$

Step 3. The desired information at ω_m is then obtained by making the following combinations

$$\underset{\sim}{H}^{(ij)}(\omega_m) = \sum_{\ell=1}^{N} e^{j\omega_m t_{\ell+1}} \underset{\sim}{\tilde{\Sigma}}_{\ell}^{(j)} + \sum_{k=1}^{N}\left[\frac{t_{k+1}-t_k}{T} - \nu_k(\omega_m)\right] \underset{\sim}{\tilde{\sigma}}_k^{(j)}$$

$$X^{(ij)}(\omega_m,\omega_m+n\omega_s) = \sum_{\ell=1}^{N} e^{j(\omega_m+n\omega_s)t_{\ell+1}} \underset{\sim}{\tilde{\Sigma}}_{\ell}^{(j)}$$

$$+ \sum_{k=1}^{N}\left[\nu_k(-n\omega_s)-\nu_k(\omega_m)\right] e^{-n\omega_s t_{k+1}} \underset{\sim}{\tilde{\sigma}}_k^{(j)}$$

where $x^{(j)}$ is the j-th component of the vector x.

As an example of the computation of sensitivities we give the sensitivity of $H^{(ij)}(\omega_m)$ with respect to the variation of the amplification A of a voltage controlled voltage source, where the voltage in branch q is A times the voltage of node m with respect to node n

$$S_A^{H^{(ij)}}(\omega_m) = \frac{A}{H^{(ij)}(\omega_m)}\Big[\sum_{r=1}^{N} \underset{\sim}{\tilde{\Sigma}}_r^{(q)}(\underset{\sim}{\Lambda}_r^{(m)}-\underset{\sim}{\Lambda}_r^{(n)})$$

$$- \sum_{k=1}^{N}\left(\frac{t_{k+1}-t_k}{T} - \nu_k(\omega_m)\right)\sigma_k^{(q)}(\underset{\sim}{\lambda}_k^{(m)}-\underset{\sim}{\lambda}_k^{(n)})\Big]$$

Another useful sensitivity is that with respect to a parameter λ which affects all capacitors

$$S_\lambda^{H^{(ij)}}(\omega_m) = \frac{\lambda}{H^{(ij)}(\omega_m)}\Big[-\sum_{r=1}^{N} \underset{\sim}{\tilde{\Lambda}}_r^T \frac{\partial \underset{\sim}{A}_r}{\partial\lambda} \underset{\sim}{A}_r$$

$$+ \sum_{r=1}^{N-1} \underset{\sim}{\tilde{\Lambda}}_{r+1}^T \frac{\partial \underset{\sim}{A}_{r+1}}{\partial\lambda} \underset{\sim}{A}_r + e^{-j\omega_m T} \underset{\sim}{\tilde{\Lambda}}_1^T \frac{\partial A_1}{\partial\lambda} A_N$$

$$- \sum_{k=1}^{N}\left(\frac{t_{k+1}-t_k}{T} - \nu_k(\omega_m)\right)\left(\underset{\sim}{\tilde{\lambda}}_k^T \frac{\partial A_k}{\partial\lambda} \underset{\sim}{\lambda}_k\right)\Big]$$

where $\partial A_k/\partial\lambda$ is known from the design. Observe that also this sensitivity is obtained with the solutions of two sets of linear equations of ω_m (step 2).

6. Analysis applications.

6.1. Scaling of element values.

In SC analysis often small capacitor values are encountered (pF's). If one wants to study the influence of parasitic capacitors, these element values can even be much smaller. These elementvalues which are very small can cause computational problems in the matrix solution because the minimum value of the number representation of the computer can be reached. Therefore in DIANA the mean value is taken of all capacitors and called $1/\alpha$. According to corollary 1 of (3° Vandewalle, 1981) all capacitorvalues and the controlling factors of VCQS may be multiplied by α and the controlling factors of QCVS may be devided by α. Now all the charges are multiplied by α but the voltages remain the same.

6.2. Applications.

The above described analysis procedure has been implemented into the DIANA program. A considerable timesaving could be achieved by using the direct frequency method compared to the previously implemented time domain approach of DIANA (3° De Man, 1980). In the time domain approach one or more impulsresponses are calculated and transformed to the frequency domain by FFT(s). Table 2. shows the results for both applied to the filter of fig. 4. First for a frequency domain analysis by both methods over 1024 points and afterwards for a sensitivity analysis of all VDV's and a transferfunction analysis. N is the resulting matrix dimension after compaction. Notice that the simulation times are improved by a factor of 6...7.
Fig. 5. shows the resulting amplitude and phase of the transferfunction on a logarithmic frequency scale.

ANALYSIS	TRANSFER FUNCTION		TRANSFER, SENS	
DIANA simulation mode	time	frequency	time	frequency
READIN(s)	1.3	1.4	1.3	1.4
SUMMARY(s)	0.4	0.4	0.5	0.4
ORDER (+COMPACTION)(s)	0.2	2.0	0.2	1.7
TRANSIENT, FREQUENCY(s)	92.8	17.1	190.3	55.2
FFT(s)	27.8		213.7	
TOTAL(s)	123.4	22.0	406.8	59.6
N	22	8	22	18

Table 2. VAX RUN STATISTICS.

The CPU time improvement is even more considerable if fewer frequency points are to be calculated. Normally one is only interested in a limited frequency range (e.g. pass band) which reduces the number of frequency points to be calculated.

A DIANA-inputdescription of the lowpass filter of fig. 4. is shown below in fig. 6. The major change compared to the time domain version is the inclusion of a .SCFREQ-card.
Notice that sinx/x-effects (SINC) and continuous I/O-coupling (CONT) as well as aliasing (BAND=) can be included.
Fig. 7. shows the results for the lowpass filter. The transfercharacteristic and the sensitivities of the amplitude to resp. C12, CL2 and C2 are shown (including SINC and CONT-effects).
This took 8.9 seconds VAXCPU over 60 frequencypoints. If no sensitivity analysis is asked this time reduces to 4.9 seconds VAXCPU. Notice that this is an improvement with a factor of 25 in CPU-time compared to the case that the time domain approach of DIANA was used.

Fig. 8. shows the frequency analysis of this filter from 0 Hz till $20 \times f_s$. Observe the effect of the sinx/x-term due to the sample and hold process.

Fig. 9. shows the effect on the passband of the lowpassfilter (fig. 4) when different opamp gains A are used in the simulations.

A first order treble tone control filter with its DIANA-input is shown in fig. 10.

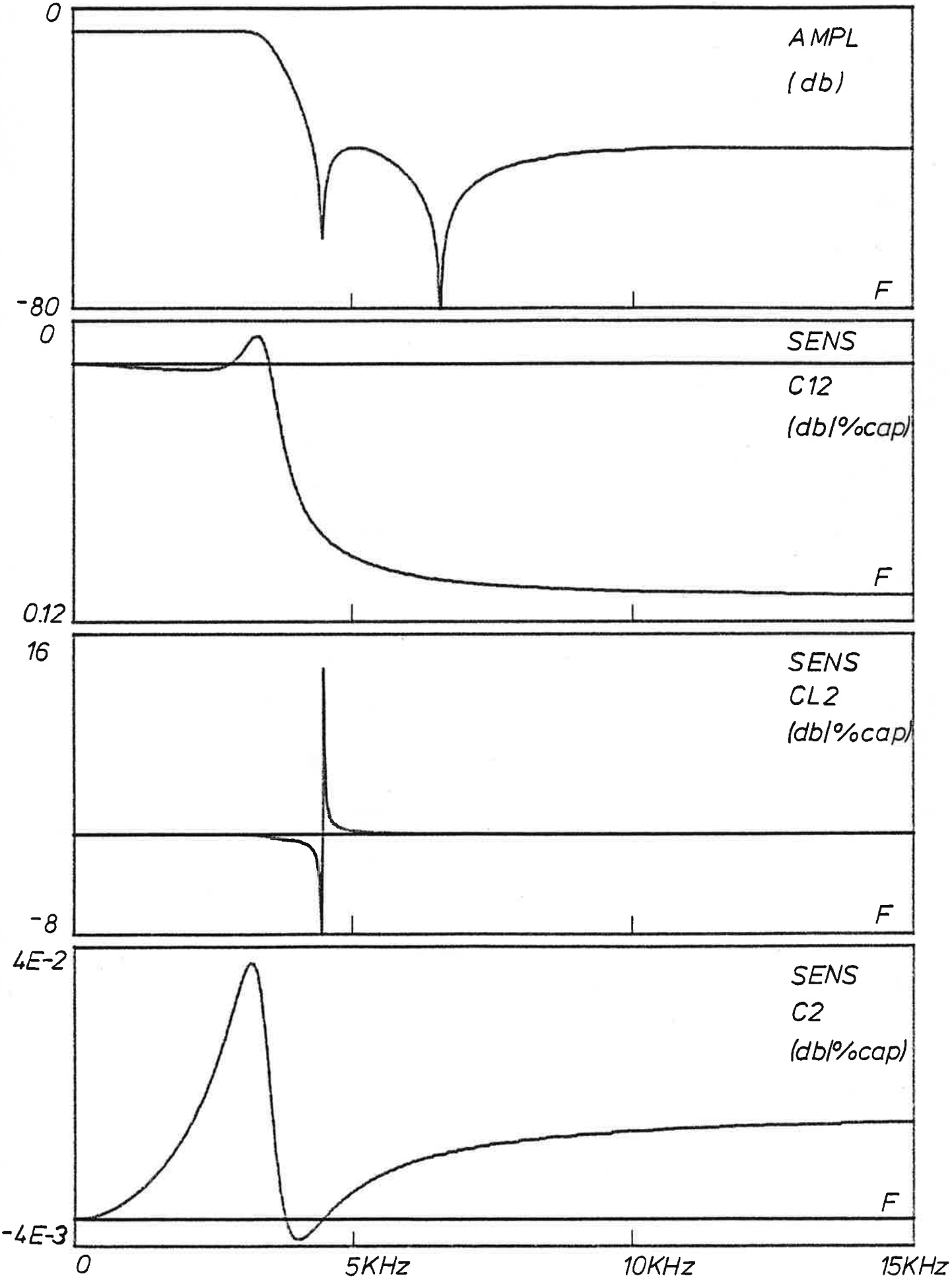

0
AMPL
(db)
-80
F
0
SENS
C12
(db/%cap)
0.12
F
16
SENS
CL2
(db/%cap)
-8
F
4E-2
SENS
C2
(db/%cap)
-4E-3
F
0
5KHz
10KHz
15KHz

fig.7.

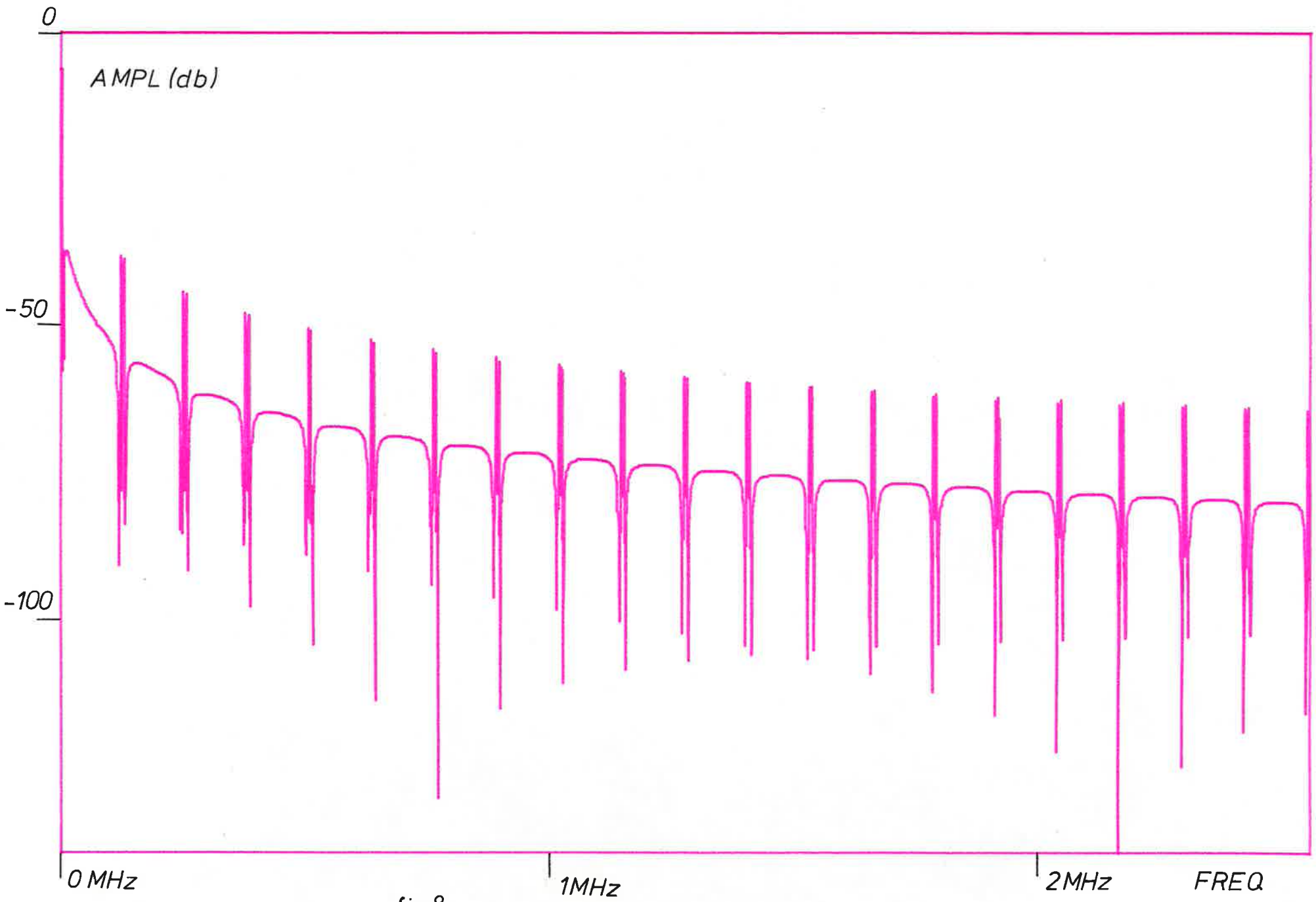


fig.8

A simulation over 60 frequency points of transfercharacteristic and sensitivities of the four capacitances took 3,5 sec VAXCPU-time. The matrix could be compacted to a dimension 10.

Fig. 11.a. shows the amplitude characteristic versus frequency for a duty cycle of 50%. Curve A is for a piecewise constant output without sample and hold effects. Curve C is with S/H effects.
Curve B obtains S/H effects as well as effects due to a continuous input-output coupling. The dotted curves A', B' and C' are the result of an aliasing analysis where an excitation at $\omega = \omega_m + \omega_s$ has an effect in the baseband at pulsation ω_m (folding effect).
This can be important to design e.g. anti-aliasing pre-filters. Fig. 11.b. shows the sensitivity analysis characteristics for the four capacitors.

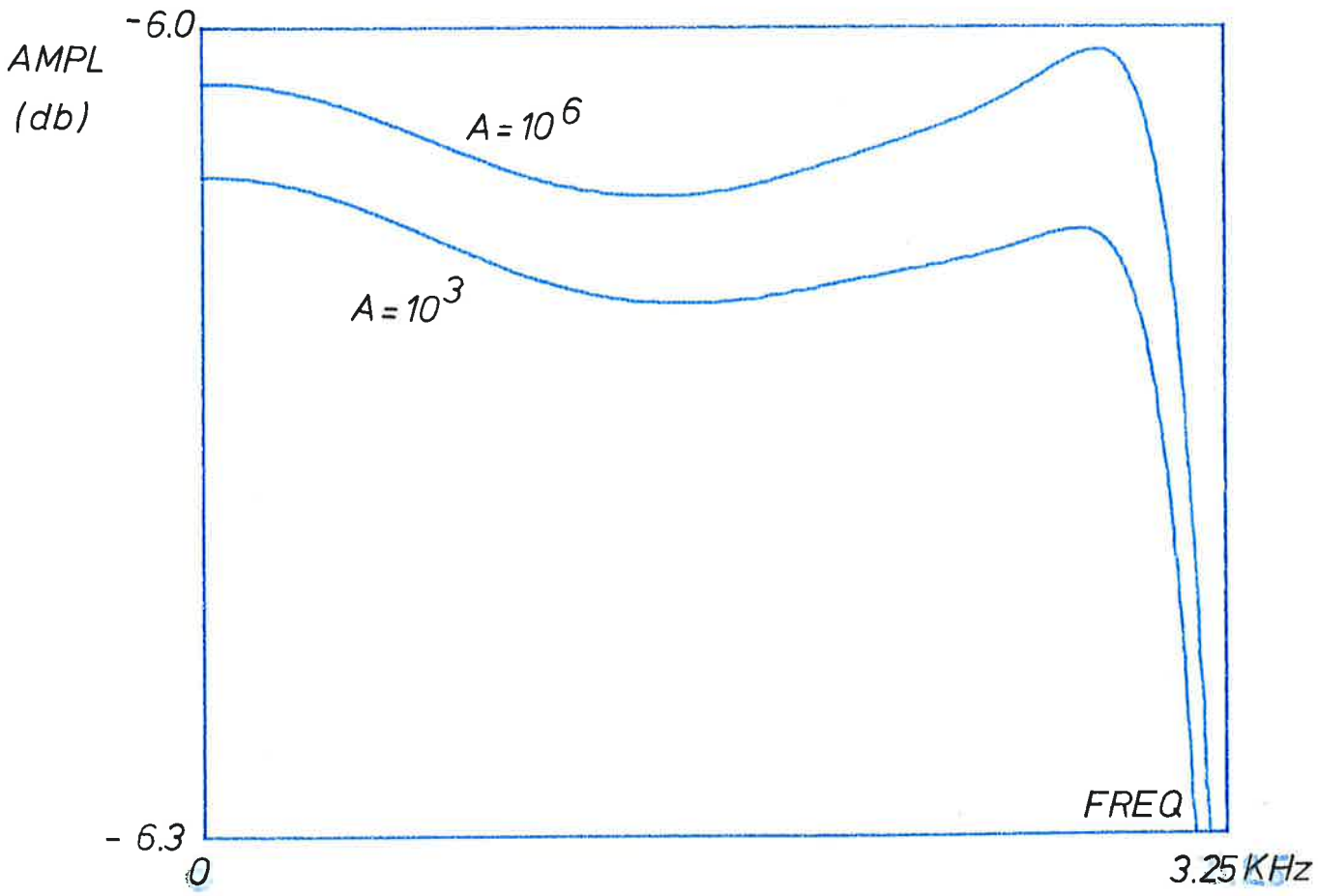
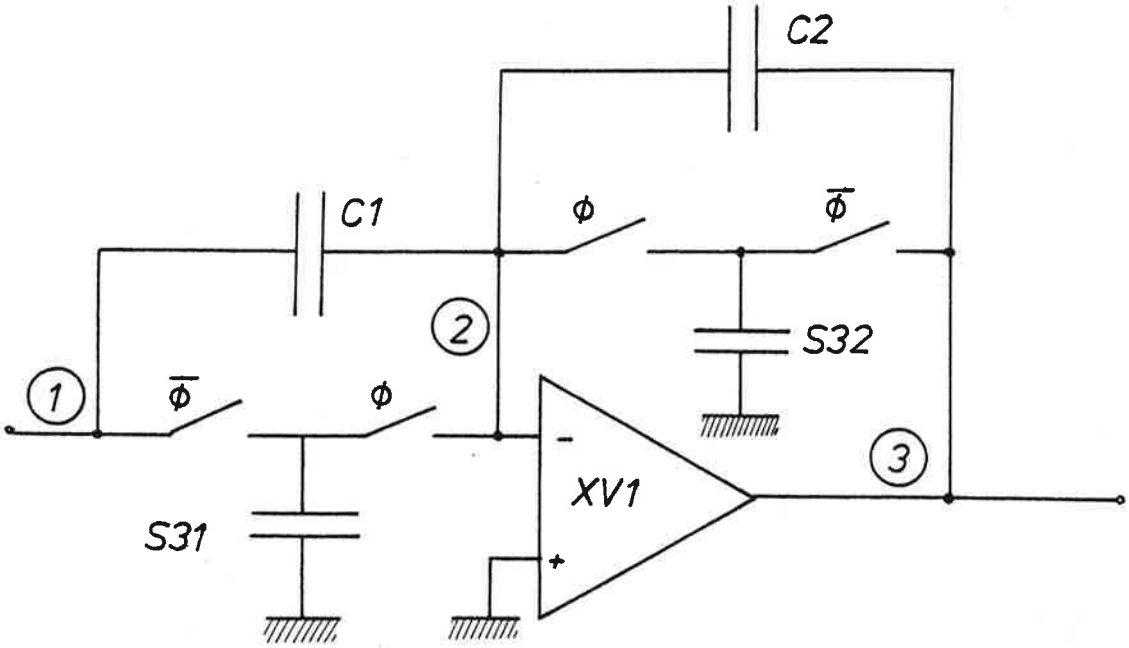


fig.9.




```
ESAT   TREBLE TONE CONTROL FILTER
.SENS CAP=S31 S32 C1 C2
.SCFREQ NLIN=60 FSTART=0.1HZ FSTOP=50KHZ
INPUT CL V-T=0 0 2 5U 0 10U CYC
PRINT 3
VIN 1 0 1
S31 0 2 1 CL 0 VT=1 C=1P
S32 0 2 3 CL 0 VT=1 C=1P
C1 1 2 7.353P
C2 2 3 1.739P
XV1 3 0 0 2 1K
.END
```

fig. 10

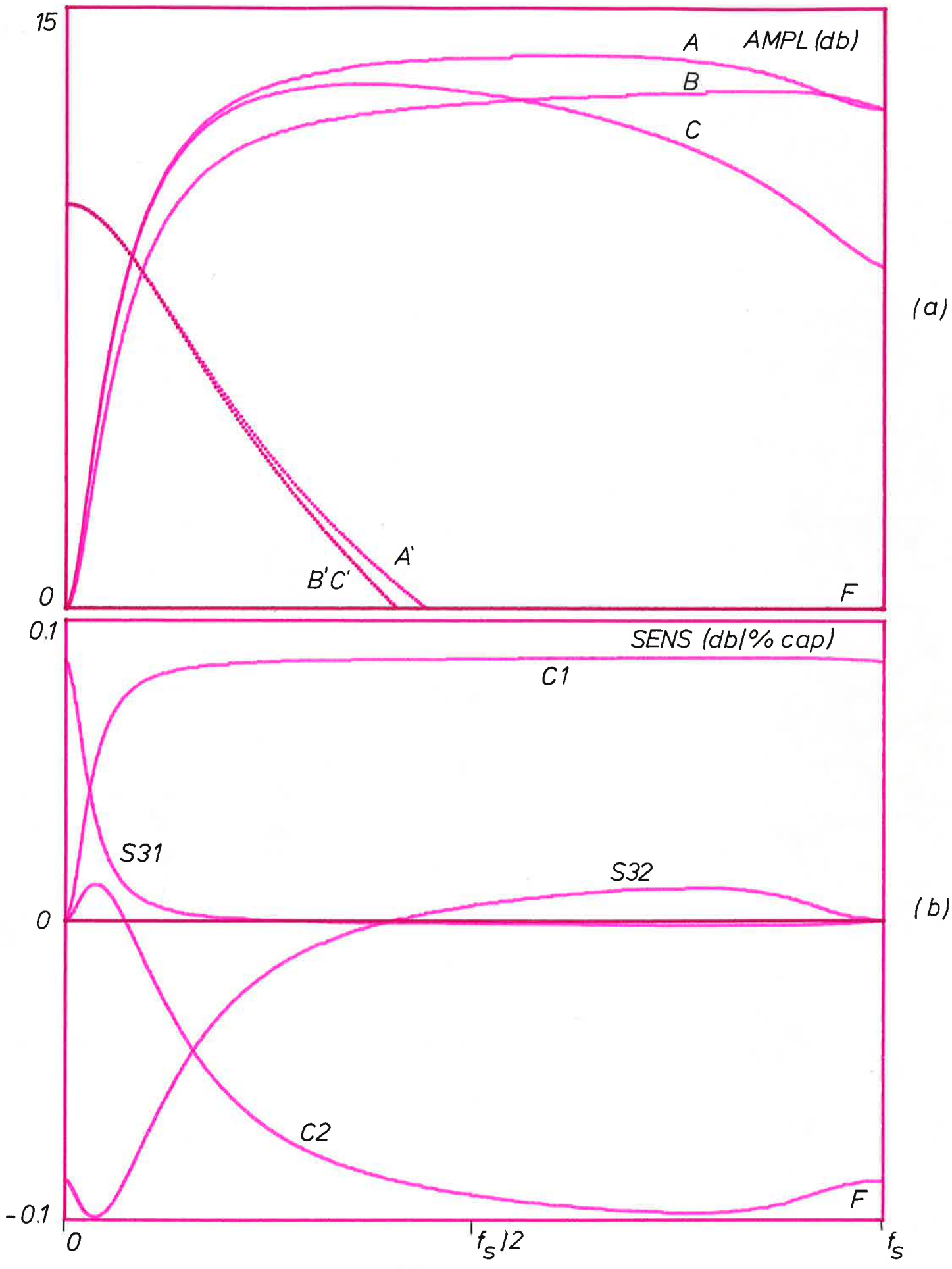


fig. 11.

7. Conclusions.

In this lecture we presented how switched capacitor circuits can be efficiently analysed by use of the z-domain MNA-matrix formalism. The z-domain MNA-formalism and its setup were studied. The compaction of the initial MNA-matrix, the use of the adjoint network and the efficient organisation of the calculations resulted in a handsome tool for the computation of sensitivities, frequency and aliasing characteristics. No constraints on the number of phases, duty cycles and continuous I/O-coupling were imposed.

Moreover this new z-domain analysis part is used as a basic building block in the analysis of SC-circuits with parasitic effects due to switch resistances and finite opamp bandwidths (Rabaey, 1981). This is discussed in the next lecture.

The computations presented here are simpler, more accurate and more flexible (any ω_m is allowed). Indeed the technique via time-domain requires the computation of several impulse responses of the original and the ajoint network until it died out and a discrete Fourier transform (DFT) of these signals (many signals in the sensitivity analysis). The accuracy of the smaller components in the DFT of selective circuits can clearly be quite low since those components are at the same level as the rounding errors. Considerable improvements in speed have been obtained in the implementation in the DIANA program with respect to the time domain version.

Appendix 1. Independence from λ in the compaction proces.

Consider a matrix equation :

$$(\underset{\sim}{A}+\lambda\underset{\sim}{B})\underset{\sim}{X} = \underset{\sim}{Y} \qquad (1)$$

where $\underset{\sim}{Y}$ is the excitation vector, $\underset{\sim}{X}$ is the unknown vector and $\underset{\sim}{A}$ is a non-singular (nxn) matrix and $\underset{\sim}{B}$ is a (nxn) matrix that may be singular. λ is a complex parameter.

1.a. Definition : λ -independent row.

A row i of $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{Y}$ in (1) is said to be λ-independent if the excitation $y_i=o$ and if there is a γ_i independent of λ such that for all j=1 ... n :

$$b_{ij} = \gamma_i\, a_{ij} \qquad (j=1...n)$$

This implies that in this equation $(1+\lambda.\gamma_i)$ is a common factor when the remaining vector does not depend on λ and can be divided.

1.b. Definition : λ-independent column.

A column j of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ in (1) is said to be λ - independent if the variable x_j is of no interest and if for all elements a_{ij} (i=1...n) of $\underset{\sim}{A}$ and all elements b_{ij} (i=1...n) of $\underset{\sim}{B}$ holds that :

$$b_{ij} = \gamma_j\, a_{ij} \qquad (i=1...n)$$

This implies that $(1+\lambda\gamma_i)$ is a common factor of the variable x_j. Therefore x_j can be scaled by $(1+\lambda\gamma_i)$.

Observe that the factors γ_i (and γ_j) in definition 1.a. and 1.b. can not be infinity because this would imply that all a_{ij} of row i (resp. column j) are 0. This is not allowed because the matrix $\underset{\sim}{A}$ is assumed to be nonsingular.

Theorem 1.

If a row i is λ-independent and if all elements b_{ij} (j=1...n) of B are put equal to zero then the resulting equation :

$$(\underset{\sim}{A} + \lambda\underset{\sim}{B}')\ \underset{\sim}{X} = \underset{\sim}{Y} \qquad (2)$$

remains equivalent to (1).

Proof : row i in matrix equation (1) is given by :

$$\sum_{j=1}^{n} a_{ij}x_j + \lambda \sum_{j=1}^{n} b_{ij}x_j = y_i \qquad (3)$$

But because row i is λ-independent $y_i=o$ and $b_{ij}= \gamma_i\, a_{ij}$.
Equation (3) can be replaced by :

$$(1+\lambda\gamma_i) \sum_{j=1}^{n} a_{ij}\, x_j = o \qquad (4)$$

Dividing (4) by $(1+\lambda\gamma_i)$ remains :

$$\sum_{j=1}^{n} a_{ij}\, x_j = o$$

which is the same as (3) but where all the b_{ij} terms (j=1...n) are put equal to zero. □

Theorem 2.

If a column j is λ-independent, and if all elements b_{ij} (i=1...n) of $\underset{\sim}{B}$ are put equal to zero then the resulting equation :

$$(\underset{\sim}{A} + \lambda\underset{\sim}{B}')\ \underset{\sim}{X}' = \underset{\sim}{Y}$$

remain equivalent to (1). In $\underset{\sim}{X}'$ $x_j=x_j\ (1+\lambda\gamma_j)$

Proof : Denote $\sum_{l=1}^{n,\,j} a_l = (\sum_{l=1}^{n} a_l) - a_j$ as a summation over all a_l except a_i. Equation (1) can now be written as :

$$\sum_{l=1}^{n} (a_{kl} + b_{kl})\, x + (a_{kj} + b_{kj})\, x_j = y_k$$

$$(k = 1...n) \qquad (5)$$

Now, if column j is λ-independent it holds that :

$$b_{kj} = \gamma_j . a_j \qquad (k = 1...n) \qquad (6)$$

Fillin (6) in (5) and replace x_j by $x_j'/(1+\lambda\gamma_j)$.
From this we obtain :

$$\sum_{l=1}^{n} (a_{kl} + b_{kl})\, x_l + a_{kj} \cdot x'_j = y_k \qquad (k = 1...n)$$

In this equivalent matrix representation we notice that all b_{ij}-terms are equal to zero. □

Theorem 3.

All λ-independent rows of (1) remain λ-independent if Gauss elimination is applied on one of these λ-independent rows.

Proof :

Suppose row i and row k to be λ-independent.

Then according to theorem 1 they can be written as :

$$\sum_{j=1}^{n} a_{ij}\, x_j = o \qquad (7)$$

$$\sum_{j=1}^{n} a_{kj} \cdot x_j = o \qquad (8)$$

If a variable x_ℓ by use of equation (7) ($a_{i\ell} \neq o$) is Gauss eleminated in the remaining equations, λ-independent row like (8) is replaced by :

$$\sum_{j=1}^{n}{}^{\ell} \left(a_{kj} - \frac{a_{k\ell}}{a_{i\ell}}\, a_{ij}\right) x_j = o \qquad (9)$$

According to definition 1, this row k is λ-independent with $\gamma_k = o$. □

Theorem 4.

All λ-independent columns of (1) remain λ-independent of an unknown variable corresponding to a λ-independent column is Gauss-eliminated.

Proof :

Suppose column j and column ℓ to be λ-independent.

According to theorem 2. the equations can then be written as :

$$\sum_{m=1}^{n}{}^{j,\ell} (a_{km} + b_{km}) x_m + a_{kj}\, x_j' + a_{k\ell} x_\ell' = y_k \qquad (k=1\ldots n)$$

Suppose that unknown variable x_j will be eliminated by use of equation i ($a_{ij} \neq o$). The remaining equations are now replaced by :

$$\sum_{m=1}^{n}{}^{j,\ell} \left[\left(a_{km} - \frac{a_{kj}\, a_{im}}{a_{ij}}\right) + \lambda \left(b_{km} - \frac{b_{kj}\, b_{im}}{ij}\right) \right] x_m +$$

$$\left(a_{k\ell} - \frac{a_{kj}\, a_{i\ell}}{a_{ij}}\right) x_\ell' = \left(1 - \frac{a_{kj}}{a_{ij}}\right) y_k \qquad (k = 1\ldots n, \text{ and } k \neq j) \qquad (10)$$

Application of definition 2 tells us that the early λ-independent column ℓ is still λ-independent with $\lambda = 0$. □

Remark : Notice that the Gauss elimination of λ-independent rows does not guarantee that λ-independent columns keep this property and vice versa.

Theorem 5.

A Gauss elimination of a λ-independent row or a λ-independent column in equation (1) results in an equation of the form :

$$(\underset{\sim}{A}' + \lambda \underset{\sim}{B}')\ \underset{\sim}{X}' = \underset{\sim}{Y}'$$

Proof : The proof of this theorem is analoguous to the proof of theorem 3 and theorem 4.□