Wireless Communication Systems
@CS.NCTU

Lecture 6: Multiple-Input Multiple-Output (MIMO)

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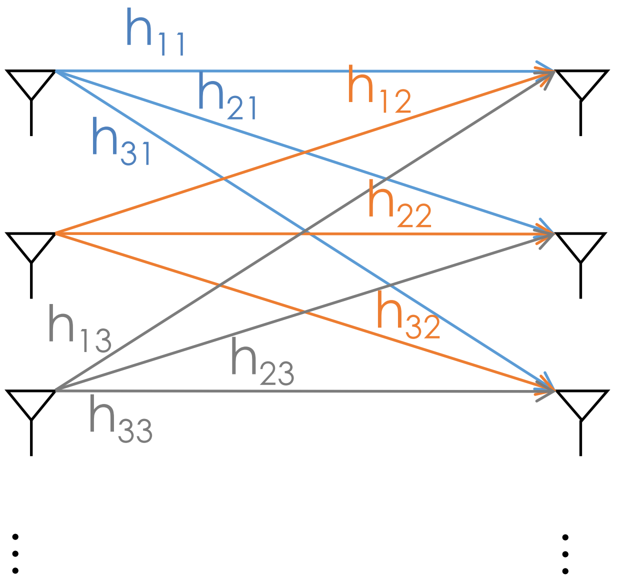
Agenda

- Channel model
- MIMO decoding
- Degrees of freedom
- Multiplexing and Diversity

MIMO

- Each node has multiple antennas
  - Capable of transmitting (receiving) multiple streams concurrently
  - Exploit antenna diversity to increase the capacity

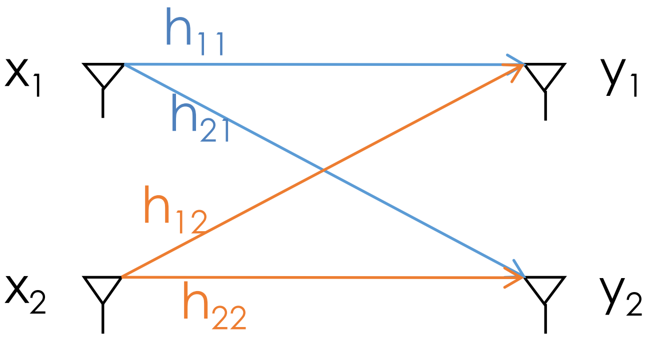


$$\mathbf{H}_{N \times M} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

N: number of antennas at Rx
M: number of antennas at Tx
H_{ij}: channel from the j-th Tx antenna to the i-th Rx antenna

Channel Model (2x2)

- Say a 2-antenna transmitter sends 2 streams simultaneously to a 2-antenna receiver



Equations

$$y_1 = h_{11}x_1 + h_{12}x_2 + n_1$$

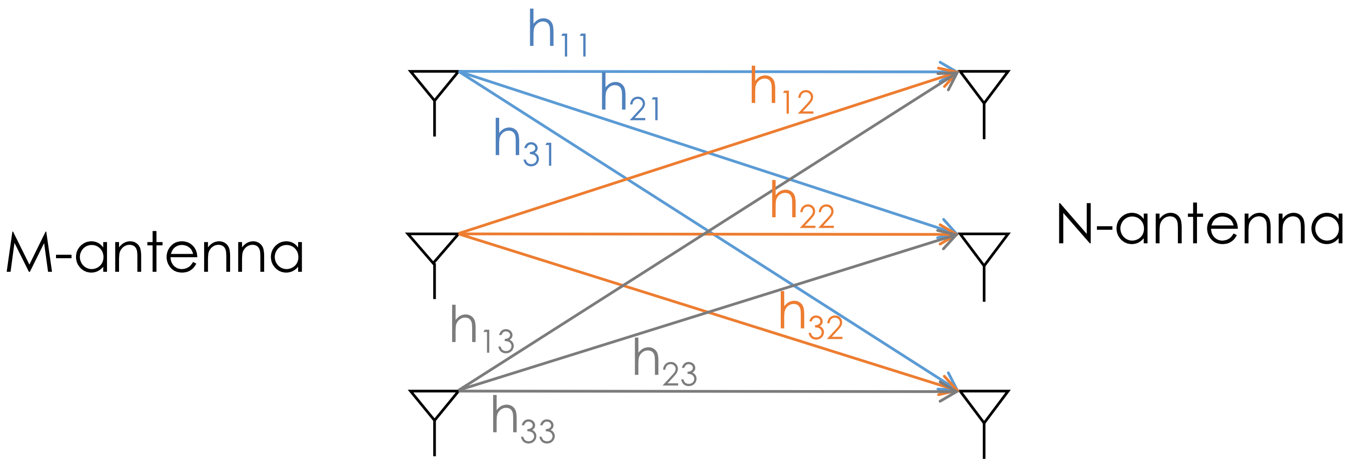
$$y_2 = h_{21}x_1 + h_{22}x_2 + n_2$$

Matrix form: $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

MIMO (MxN)

- An M-antenna Tx sends to an N-antenna Rx



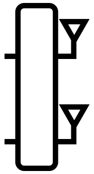
$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1M} \\ h_{21} & h_{22} & \cdots & h_{2M} \\ \vdots & & \ddots & \\ h_{N1} & h_{N2} & \cdots & h_{NM} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{pmatrix}$$

Antenna Space (2x2, 3x3)

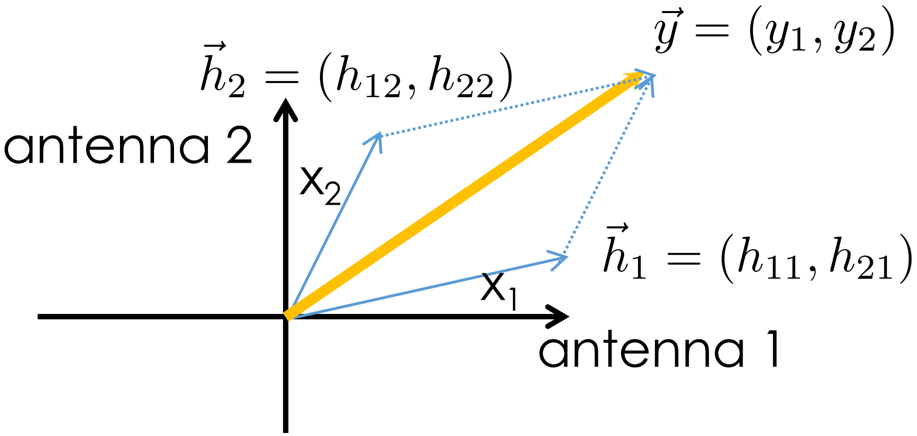
N-antenna node receives in N-dimensional space

2 x 2

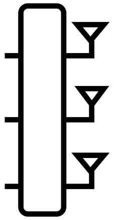


$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} x_1 + \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} x_2 + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

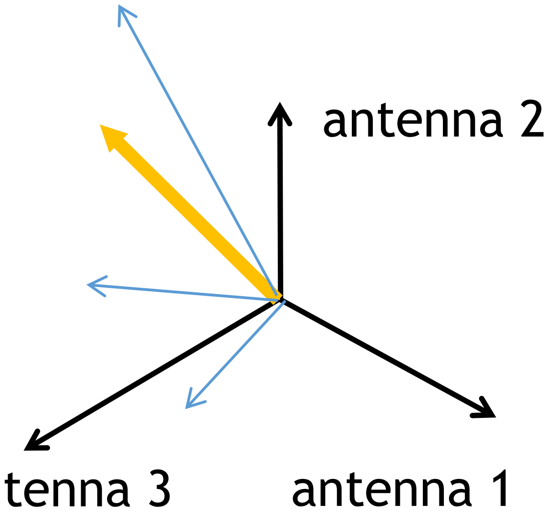
$$\vec{y} = \vec{h}_1 x_1 + \vec{h}_2 x_2 + \vec{n}$$



3 x 3



$$\vec{y} = \vec{h}_1 x_1 + \vec{h}_2 x_2 + \vec{h}_3 x_3 + \vec{n}$$



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- Channel model
- MIMO decoding
- Degrees of freedom
- Multiplexing and Diversity

Zero-Forcing (ZF) Decoding

- Decode x_1

orthogonal vectors

$$+\Big)\ \underline{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} x_1 + \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} x_2 + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}} \quad \begin{array}{l} * \ h_{22} \\ * \ - h_{12} \end{array}$$

$$y_1 h_{22} - y_2 h_{12} = (h_{11} h_{22} - h_{21} h_{12}) x_1 + n'$$

$$\begin{aligned} x_1' &= \frac{y_1 h_{22} - y_2 h_{12}}{h_{11} h_{22} - h_{21} h_{12}} \\ &= x_1 + \frac{n'}{h_{11} h_{22} - h_{21} h_{12}} \\ &= x_1 + \frac{n'}{\vec{h}_1 \cdot \vec{h}_2^{\perp}} \end{aligned}$$

Zero-Forcing (ZF) Decoding

- Decode x_2

orthogonal vectors

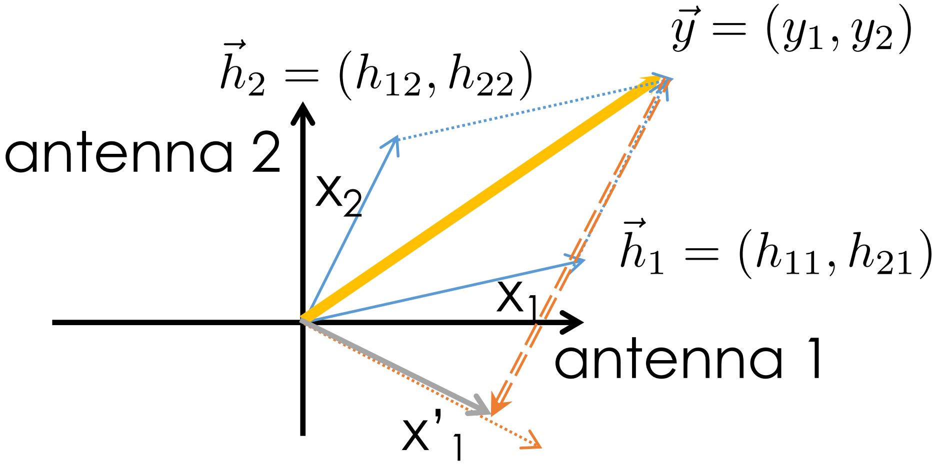
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} x_1 + \begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} x_2 + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \qquad \begin{array}{l} * \ h_{21} \\ * \ \text{-}\ h_{11} \end{array}$$

$$+)\ \underline{\hspace{20em}}$$

$$y_1 h_{21} - y_2 h_{11} = (h_{12} h_{21} - h_{22} h_{11}) x_2 + n'$$

$$\begin{aligned} x_2' &= \frac{y_1 h_{21} - y_2 h_{11}}{h_{12} h_{21} - h_{22} h_{11}} \\ &= x_2 + \frac{n'}{h_{12} h_{21} - h_{22} h_{11}} \\ &= x_2 + \frac{n'}{\vec{h}_2 \cdot \vec{h}_1^{\perp}} \end{aligned}$$

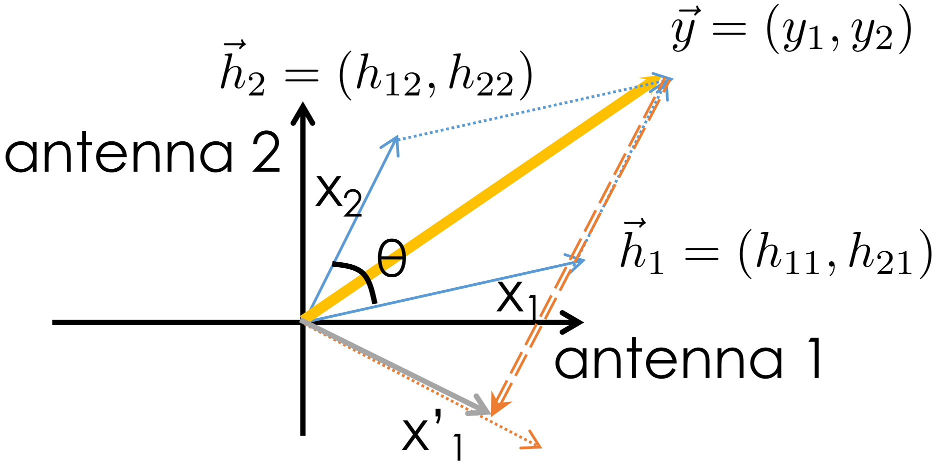
ZF Decoding (antenna space)



$|x'_1| \le |x_1|$

- To decode x_1, project the received signal y onto the interference-free direction $h_2^{\perp}$
- To decode x_2, project the received signal y onto the interference-free direction $h_1^{\perp}$
- SNR reduces if the channels h_1 and h_2 are correlated, i.e., not perfect orthogonal ($h_1 \cdot h_2=0$)

SNR Loss due to ZF Detection



$$|x'_1|^2 = |x_1|^2 \cos^2(90 - \theta) = |x_1|^2 \sin^2(\theta)$$

- From equation: $x'_1 = x_1 + \frac{n}{\vec{h}_1 \cdot \vec{h}_2^{\perp}}$

$\text{SNR}_{ZF} = \text{SNR}_{SISO}$ when $h_1 \perp h_2$

$$\text{SNR}' = \frac{|x_1|^2}{N_0/(\vec{h}_1 \cdot \vec{h}_2^{\perp})^2} = \frac{|x_1|^2 \sin^2(\theta)}{N_0} = \text{SNR} * \sin^2(\theta)$$

- The more correlated the channels (the smaller angles), the larger SNR reduction

When will MIMO Fail?

- In the worst case, SNR might drop down to 0 if the channels are strongly correlated to each other, e.g., $h_1 /\!/ h_2$ in the 2x2 MIMO

- To ensure channel independency, should guarantee the full rank of H
  - Antenna spacing at the transmitter and receiver must exceed half of the wavelength

ZF Decoding – General Eq.

- For a N x M MIMO system,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

- To solve **x**, find a decoder **W** satisfying the constraint

$$\mathbf{W}\mathbf{H} = \mathbf{I}, \text{ then } \mathbf{x}' = \mathbf{W}\mathbf{y} = \mathbf{x} + \mathbf{W}\mathbf{n}$$

→ **W** is the pseudo inverse of **H**

$$\mathbf{W} = (\mathbf{H}^*\mathbf{H})^{-1}\mathbf{H}^*$$

ZF-SIC Decoding

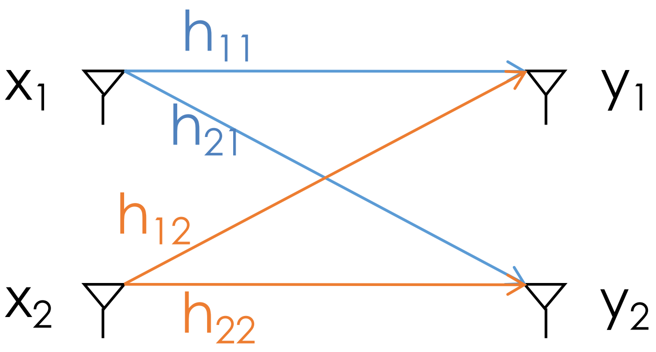
- Combine ZF with SIC to improve SNR
  - Decode one stream and subtract it from the received signal
  - Repeat until all the streams are recovered
  - Example: after decoding x_2, we have $y_1 = h_1x_1+n_1$ → decode x_1 using standard SISO decoder
- Why it achieves a higher SNR?
  - The streams recovered after SIC can be projected to a smaller subspace → lower SNR reduction
  - In the 2x2 example, x_1 can be decoded as usual without ZF → no SNR reduction (though x2 still experience SNR loss)

Other Detection Schemes

- Maximum-Likelihood (ML) decoding
  - Measure the distance between the received signal and all the possible symbol vectors
  - Optimal Decoding
  - High complexity (exhaustive search)
- Minimum Mean Square Error (MMSE) decoding
  - Minimize the mean square error
  - Bayesian approach: conditional expectation of **x** given the known observed value of the measurements
- ML-SIC, MMSE-SIC

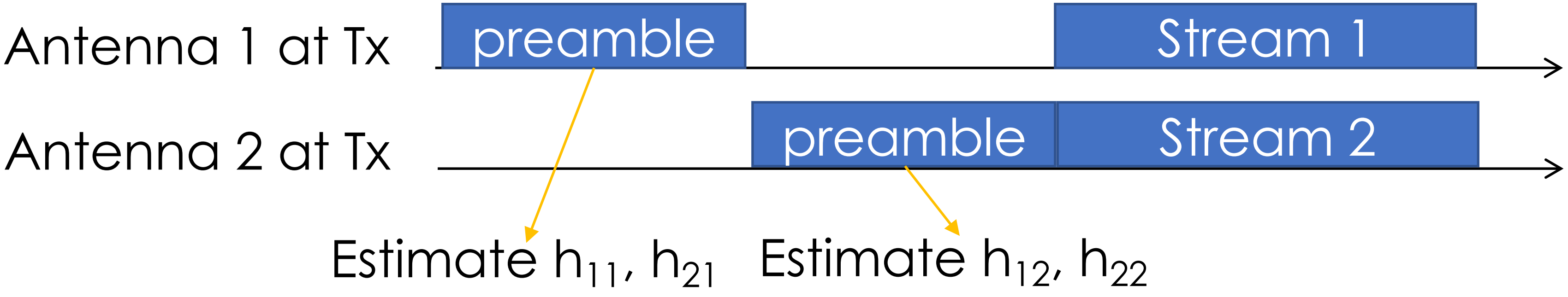
Channel Estimation

- Estimate N x M matrix H



$$\begin{aligned} y_1 &= \boxed{h_{11}} x_1 + \boxed{h_{12}} x_2 + n_1 \\ y_2 &= \boxed{h_{21}} x_1 + \boxed{h_{22}} x_2 + n_2 \end{aligned}$$

Two equations, but four unknowns



Antenna 1 at Tx: preamble, Stream 1

Antenna 2 at Tx: preamble, Stream 2

Estimate h_{11}, h_{21} Estimate h_{12}, h_{22}

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- Multiplexing and Diversity

Degree of Freedom

For N x M MIMO channel

- Degree of Freedom (DoF): min {N,M}
  - Can transmit at most DoF streams
- Maximum diversity: NM
  - There exist NM paths among Tx and Rx

MIMO Gains

- Multiplex Gain
  - Exploit DoF to deliver multiple streams concurrently
- Diversity Gain
  - Exploit path diversity to increase the SNR of a single stream
  - Receive diversity and transmit diversity

Multiplexing-Diversity Tradeoff

- Tradeoff between the diversity gain and the multiplex gain
- Say we have a N x N system
  - Degree of freedom: N
  - The transmitter can send k streams concurrently, where $k \leq N$
  - If $k < N$, leverage partial multiplexing gains, while each stream gets some diversity
  - The optimal value of k maximizing the capacity should be determined by the tradeoff between the diversity gain and multiplex gain

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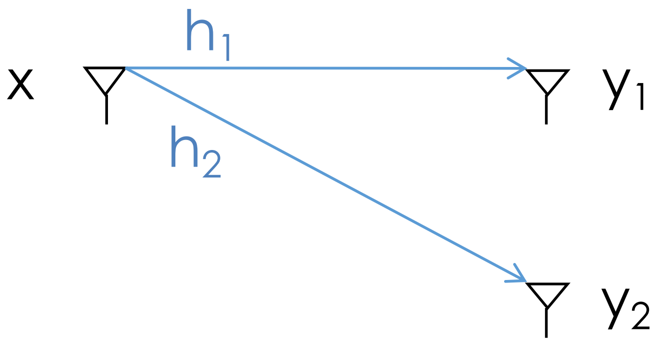
Receive Diversity

- 1 x 2 example

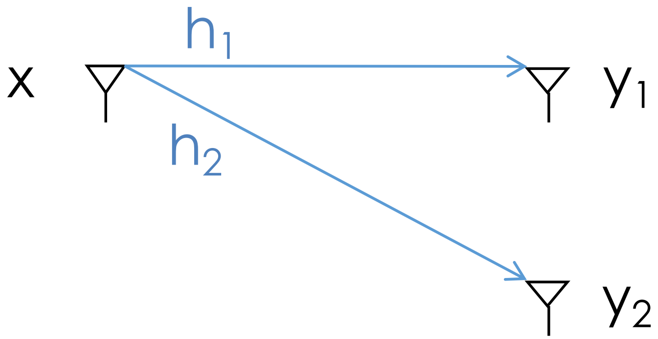


$$y_1 = h_1 x + n_1$$
$$y_2 = h_2 x + n_2$$

  - Uncorrelated whit Gaussian noise with zero mean
  - Packet can be delivered through at least one of the many diverse paths

Theoretical SNR of Receive Diversity

- 1 x 2 example



- Increase SNR by 3dB
- Especially beneficial for the low SNR link

$$\text{SNR} = \frac{P(2X)}{P(n_1 + n_2)}, \text{ where } P \text{ refers to the power}$$

$$= \frac{E[(2X)^2]}{E[n_1^2 + n_2^2]}$$

$$= \frac{4E[X^2]}{2\sigma}, \text{ where } \sigma \text{ is the variance of AWGN}$$

$$= 2 * \text{SNR}_{\text{single antenna}}$$

Maximal Ratio Combining (MRC)

- Extract receive diversity via MRC decoding
- Multiply each **y** with the conjugate of the channel

$$\begin{aligned} y_1 &= h_1 x + n_1 \\ y_2 &= h_2 x + n_2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} h_1^* y_1 &= |h_1|^2 x + h_1^* n_1 \\ h_2^* y_2 &= |h_2|^2 x + h_2^* n_2 \end{aligned}$$

- Combine two signals constructively

$$h_1^* y_1 + h_2^* y_2 = (|h_1|^2 + |h_2|^2)x + (h_1^* + h_2^*)n$$

- Decode using the standard SISO decoder

$$x' = \frac{h_1^* y_1 + h_2^* y_2}{(|h_1|^2 + |h_2|^2)} + n'$$

Achievable SNR of MRC

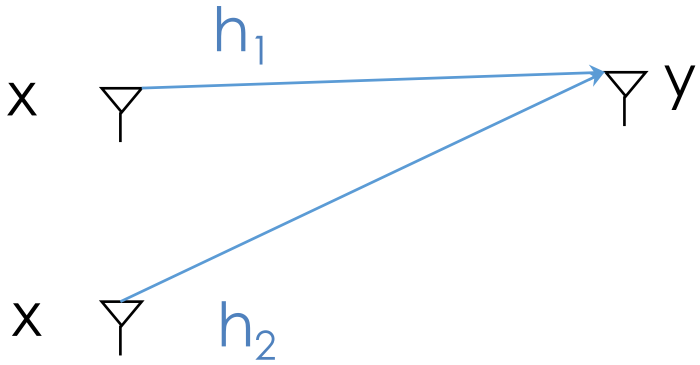
$$h_1^* y_1 + h_2^* y_2 = (|h_1|^2 + |h_2|^2)x + (h_1^* + h_2^*)n$$

$$\text{SNR}_{\text{MRC}} = \frac{E[((|h_1|^2 + |h_2|^2)X)^2]}{(h_1^* + h_2^*)^2 n^2} = \frac{(|h_1|^2 + |h_2|^2)^2 E[X^2]}{(|h_1|^2 + |h_2|^2)\sigma^2} = \frac{(|h_1|^2 + |h_2|^2) E[X^2]}{\sigma^2}$$

$$\text{SNR}_{\text{single}} = \frac{E[|h_1|^2 X^2]}{n^2} = \frac{|h_1|^2 E[X^2]}{\sigma^2}$$

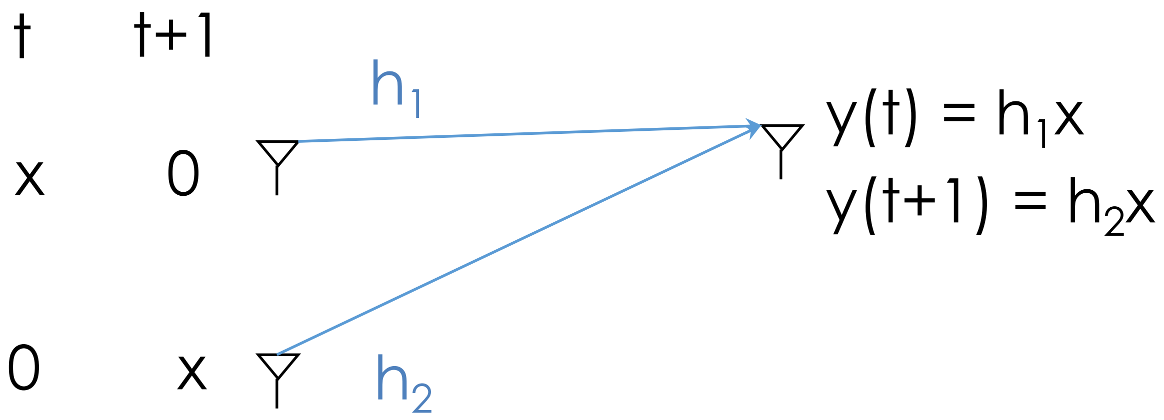
- gain $= \dfrac{|h_1|^2 + |h_2|^2}{|h_1|^2}$
- ~2x gain if $|h_1| \sim= |h_2|$

Transmit Diversity



- Signals go through two diverse paths
- Theoretical SNR gain: similar to receive diversity
- How to extract the SNR gain?
  - Simply transmit from two antennas simultaneous? ?
  - No! Again, h_1 and h_2 might be destructive

Transmit Diversity: Repetitive Code



$y(t) = h_1x$

$y(t+1) = h_2x$

- Deliver a symbol twice in two consecutive time slots
- Repetitive code

$$\mathbf{X} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

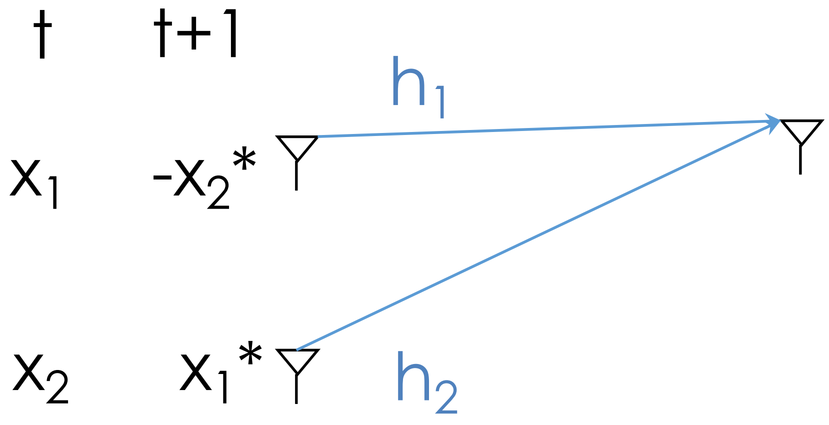
time

space

- Diversity: 2
- Data rate: 1/2 symbols/s/Hz

- Decode and extract the diversity gain via MRC
- Improve SNR, but reduce the data rate!!

Transmit Diversity: Alamouti Code



$y(t) = h_1x_1+h_2x_2^* + n$

$y(t+1) = h_2x_1^* - h_1x_2 + n$

- Deliver 2 symbols in two consecutive time slots, but switch the antennas
- Alamouti code (space-time block code)

time

$$\mathbf{x} = \begin{pmatrix} x_1 & -x_2 \\ x_2^* & x_1^* \end{pmatrix}$$

space

- Diversity: 2
- Data rate: 1 symbols/s/Hz

- Improve SNR, while, meanwhile, maintain the data rate

Transmit Diversity: Alamouti Code

y(t) = h_1x_1+h_2x_2* + n
y(t+1) = h_2x_1* - h_1x_2+ n

- Decoding

$$h_1^* y(t) = |h_1|^2 x_1 + h_1^* h_2 x_2^* + h_1^* n$$

$$y^*(t+1) = h_2^* x_1 - h_1^* x_2^* + n^*$$

$$h_2 y^*(t+1) = |h_2|^2 x_1 - h_1^* h_2 x_2^* + h_2 n^*$$

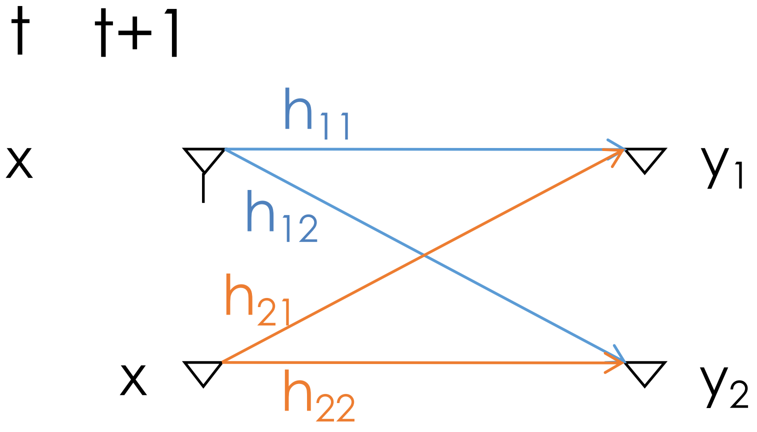
$$\Longrightarrow \quad h_1^* y(t) + h_2 y^*(t+1) = (|h_1|^2 + |h_2|^2) x_1 + h_1^* n + h_2 n^*$$

- Achievable SNR

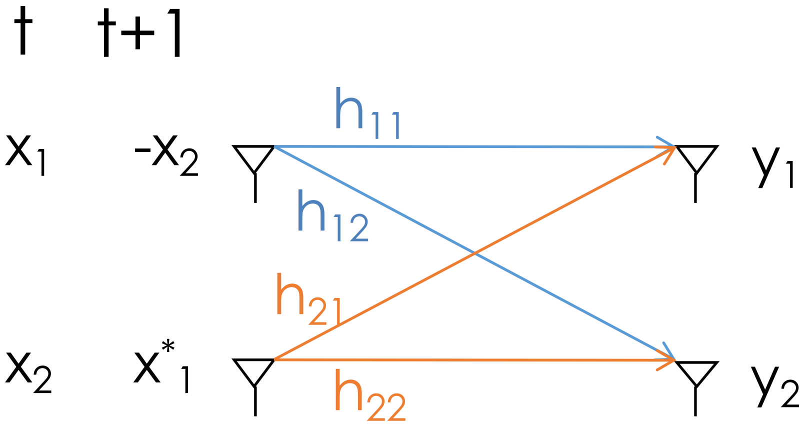
$$\frac{(|h_1|^2 + |h_2|^2)^2 E[X^2]}{(h_1^* n + h_2 n^*)}$$

$$= \frac{(|h_1|^2 + |h_2|^2)^2 E[X^2]}{(|h_1|^2 + |h_2|^2)\sigma^2} = \frac{(|h_1|^2 + |h_2|^2) E[X^2]}{\sigma^2}$$

Multiplexing-Diversity Tradeoff



Repetitive scheme



Alamouti scheme

$$\mathbf{X} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} x_1 & -x_2 \\ x_2^* & x_1^* \end{pmatrix}$$

Diversity: 4
Data rate: 1/2 sym/s/Hz

Diversity: 4
Data rate: 1 sym/s/Hz

But 2x2 MIMO has 2 degrees of freedom

Quiz

- Explain what is the channel correlation
- With ZF decoding, the more correlated the channel, the 1) higher or 2) lower the SNR?
- What is the degrees of freedom for a 8 x 6 MIMO system?