Chapter 7 z-Transform

- Introduction
- z Transform
- Unilateral z Transform
- Properties Unilateral z Transform
- Inversion of Unilateral z Transform
- Determining the Frequency Response from Poles and Zeros

PSPLAB

Introduction

- Role in Discrete-Time Systems
  - z-Transform is the discrete-time counterpart of the Laplace transform.
- Response of Discrete-Time Systems
  - If the system

    $2y[n] + 3y[n-1] + y[n-2] = u[n] + u[n-1] - u[n-2]$ for $n = 0, 1, 2$
  - The response of the system is excited by an input u[n] and some initial conditions.
  - The difference equations are basically algebraic equations, their solutions can be obtained by direct substitution.
  - The solution however is not in closed form and is difficult to develop general properties of the system.
  - A number of design techniques have been developed in the z-Transform domain.

z-Transform

- Positive and Negative Time Sequence
  - A discrete-time signal x[n] , where n is an integer ranging ($-\infty<n<\infty$), is called a positive-time sequence if x[n]=0 for n < 0; it is called a negative-time sequence if x[n] = 0 for n > 0.
  - We mainly consider the positive-time sequences.
- z-Transform Pair
  - The z-transform is defined as

$$X(z) \equiv Z[x[n]] \equiv \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

    - where z is a complex variable, called the z-transform variable.
- Example
  - x[n] = { 1, 2, 5, 7, 0, 1}; x[n] = $(1/2)^n$ u{n}

z-Transform (c.1)

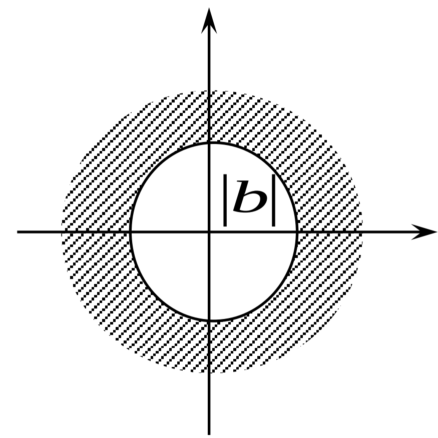
- Example
  - f[n] = b^n for all positive integer k and b is a real or complex number.

$$F(z) = \sum_{n=0}^{\infty} f[n] z^{-n} = \sum_{n=0}^{\infty} b^n z^{-n} = \sum_{n=0}^{\infty} (bz^{-1})^n$$

  - If $\left|bz^{-1}\right| < 1$, then the infinity power series converges and

$$F(z) = \frac{1}{1 - bz^{-1}} = \frac{z}{z - b}$$

  - The region $|b| < |z|$ is called the region of convergence.



- Unit Step Sequence
  - The unit sequence is defined as

$$q[n] = \begin{cases} 1 & for\ n = 0,1,2, \\ 0 & for\ n < 0 \end{cases}$$

  - The z-Transform is

$$Q(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}$$

- Exponential Sequence
  - f[n] = e ^{anT}

$$F(z) = \sum_{n=0}^{\infty} e^{anT} z^{-n} = \frac{1}{1 - e^{aT} z^{-1}}$$

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z-Transform (c.2)

- Region of Convergence
  - For any given sequence, the set of values of z for which the z-transform converges is called the region of convergence.
- Viewpoints
  - The representation of the complex variable z

$$z = re^{j\omega}$$

  - Consider the z-transform

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

  - Convergent Condition

$$\sum_{n=-\infty}^{\infty} \left| x[n]r^{-n} \right| < \infty$$

ROC includes the unit circle
==> Fourier Transform converges

Convergence of the z-Transform
==> The z-transform and its derivatives must be continuous function of z.

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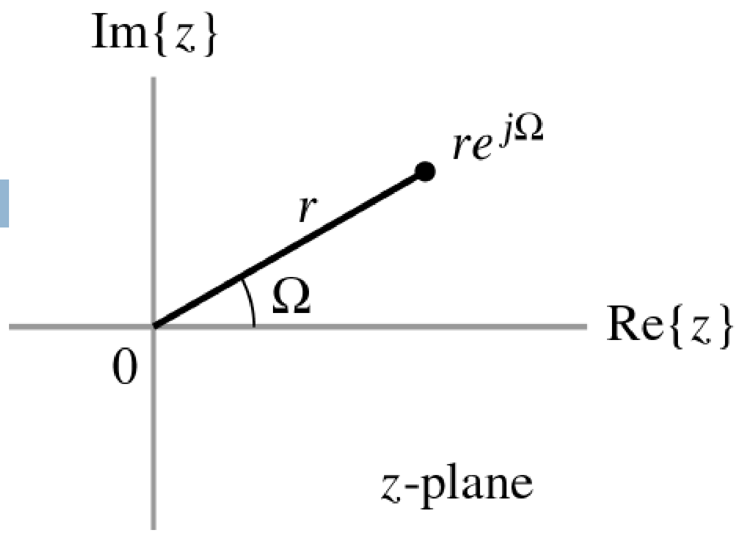
Unilateral z Transform



7.2.2 *The z-Plane*

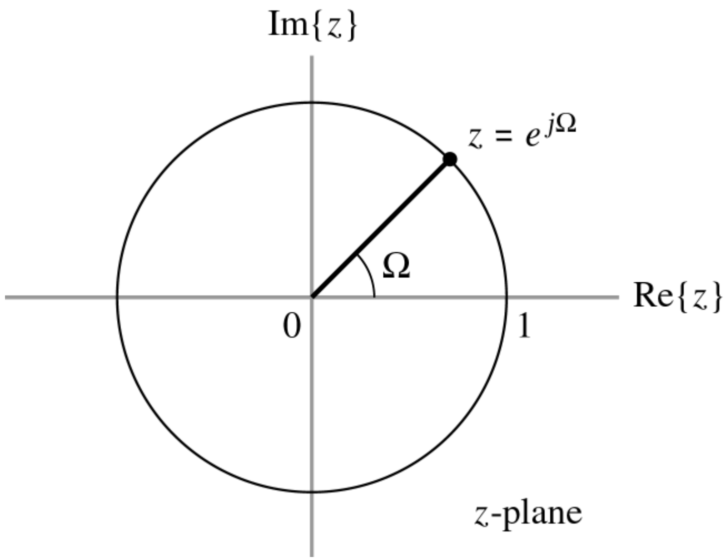
2. If $x[n]$ is absolutely summable, then the DTFT obtained from the z-transform by setting $r = 1$, or substituting $z = e^{j\Omega}$ into Eq. (7.4). That is,

$$X(e^{j\Omega}) = X(z)|_{z=e^{j\Omega}}. \qquad (7.6)$$

3. The equation $z = e^{j\Omega}$ describes a circle of unit radius centered on the origin in the z-plane.

Fig. 7.4.

Unit circle in the z-plane.



♣ The frequency Ω in the DTFT corresponds to the point on the unit circle at an angle Ω with respect to the positive real axis.

Unilateral z Transform

Determine the *z*-transform of the signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Use the *z*-transform to determine the DTFT of *x*[*n*].

<Sol.>

1. We substitute the prescribed *x*[*n*] into Eq. (7.4) to obtain

$$X(z) = z + 2 - z^{-1} + z^{-2}.$$

2. We obtain the DTFT from *X*(*z*) by substituting $z = e^{j\Omega}$:

$$X(e^{j\Omega}) = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}.$$

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Unilateral z Transform

7.2.3 *Poles and Zeros*

1. The *z*-transform in terms of two polynomials in z^{-1}:

$$X(z) = \frac{b_0 + b_1 z_{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots a_N z^{-N}}.$$

where $\tilde{b} = b_0 / a_0 \equiv \text{gain} \quad \text{factor}$

$$\Rightarrow \quad X(z) = \frac{\tilde{b} \prod_{k=1}^{M} \left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - d_k z^{-1}\right)}.$$

2. The c_k are the roots of the numerator polynomial $\Rightarrow$ the zeros of $X(z)$.
The d_k are the roots of the denominator polynomial $\Rightarrow$ the poles of $X(z)$.

3. Symbols in the z-plane:

$\times \;\Rightarrow\;$ poles; $\;\bigcirc \;\Rightarrow\;$ zeros

Unilateral z Transform

Example 7.2 ***z-TRANSFORM OF A CAUSAL EXPONENTIOAL SIGNAL***

Determine the *z*-transform of the signal

$$x[n] = \alpha^n u[n].$$

Depict the ROC and the location of poles and zeros of *X*(*z*) in the z-plane.

<Sol.>

1. Substituting $x[n] = \alpha^n u[n]$ into Eq. (7.4) yields

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n.$$

2. This is a geometric series of infinite length in the ratio α/z; the sum converges, provided that $|\alpha/z| < 1$, or $|z| > |\alpha|$. Hence,

$$\begin{aligned} X(z) &= \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha| \\ &= \frac{z}{z-\alpha}, \quad |z| > |\alpha|. \end{aligned} \qquad (7.7)$$

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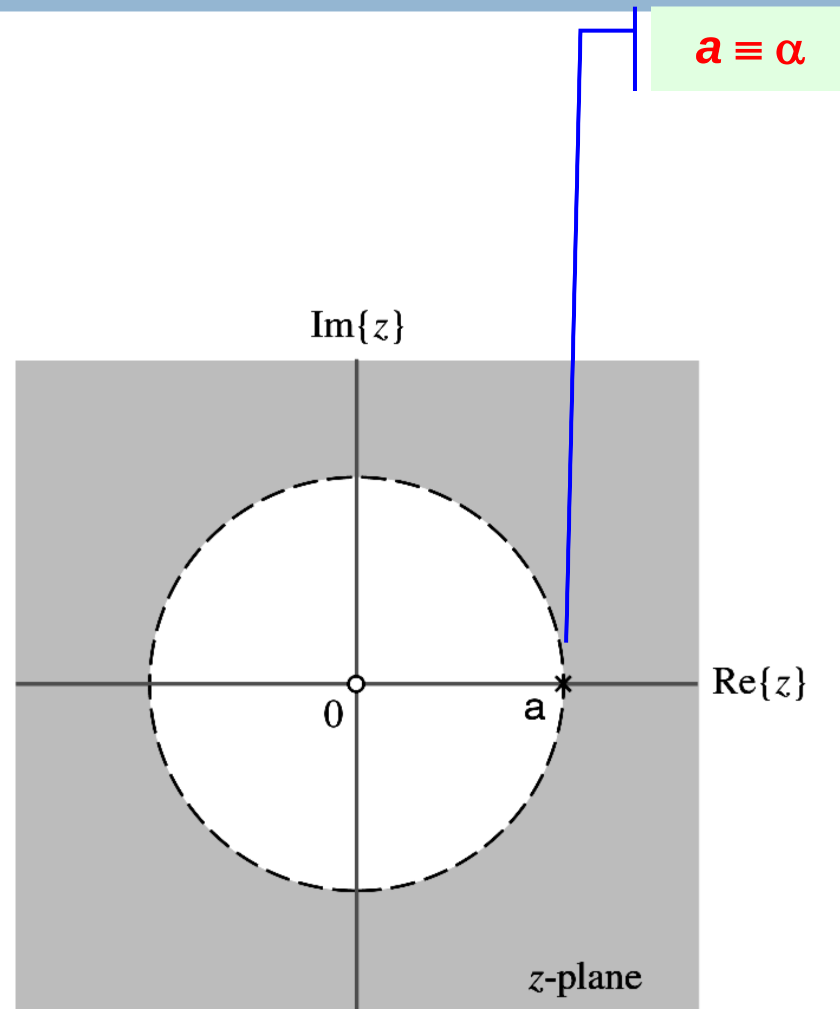
Unilateral z Transform

3. There is thus a pole at $z = \alpha$ and a zero at $z = 0$, as illustrated in Fig 7.5. the ROC is depicted as the shaded region of the z-plane.



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Unilateral z Transform

Example 7.3 ***z-TRANSFORM OF A ANTICAUSAL EXPONENTIOAL SIGNAL***

Determine the *z*-transform of the signal

$$y[n] = -\alpha^n u[-n-1].$$

Depict the ROC and the location of poles and zeros of *X*(*z*) in the *z*-plane.

<Sol.> **1. We substituting $y[n] = -\alpha^n u[-n-1]$ into Eq.(7.4) and write**

$$Y(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n$$

$$= -\sum_{k=1}^{\infty} \left(\frac{z}{\alpha}\right)^k$$

$$= 1 - \sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k.$$

2. The sum converges, provide that $|z/\alpha| < 1$, or $|z| < |\alpha|$. Hence,

$$Y(z) = 1 - \frac{1}{1 - z\alpha^{-1}}, \quad |z| < |\alpha|,$$

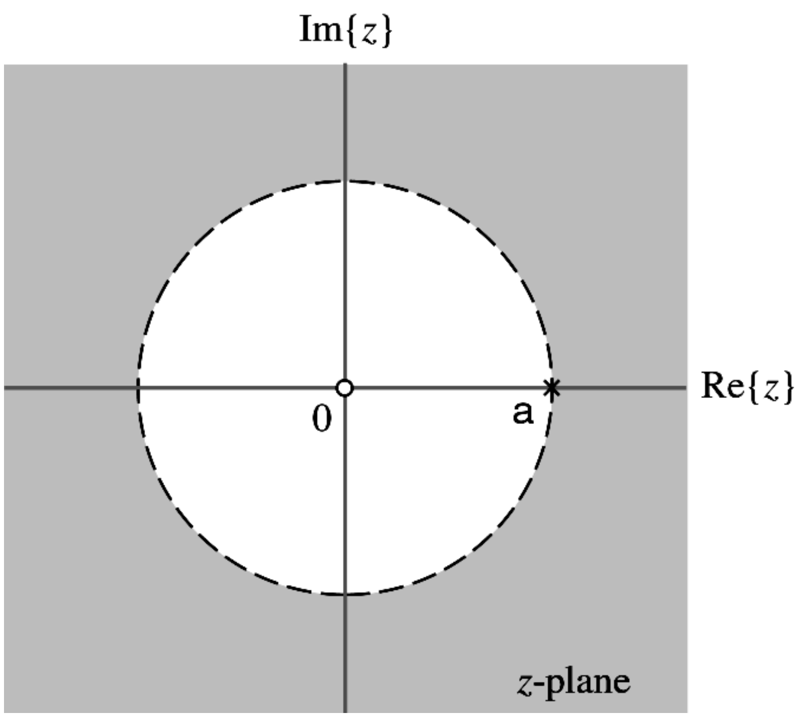
$$= \frac{z}{z-\alpha}, \quad |z| < |\alpha| \qquad (7.8)$$

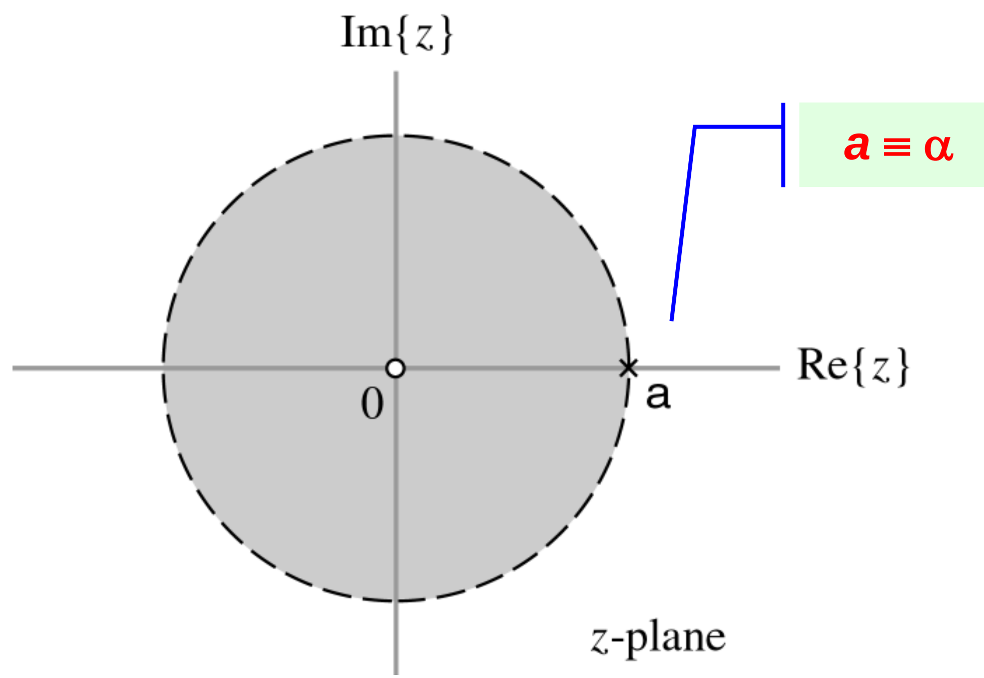
3. The ROC and the location of poles and zero are depicted in Fig 7.6.

Unilateral z Transform

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$
$$= \frac{z}{z-\alpha}, \quad |z| > |\alpha|.$$

$$Y(z) = 1 - \frac{1}{1-z\alpha^{-1}}, \quad |z| < |\alpha|,$$
$$= \frac{z}{z-\alpha}, \quad |z| < |\alpha|$$





Unilateral z Transform

Example 7.4 *z-TRANSFORM OF A TWO SIDED SIGNAL*

Determine the *z*-transform of

$$x[n] = -u[-n-1] + \left(\frac{1}{2}\right)^n u[n].$$

Depict the ROC and the location of poles and zeros of *X*(*z*) in the *z*-plane.

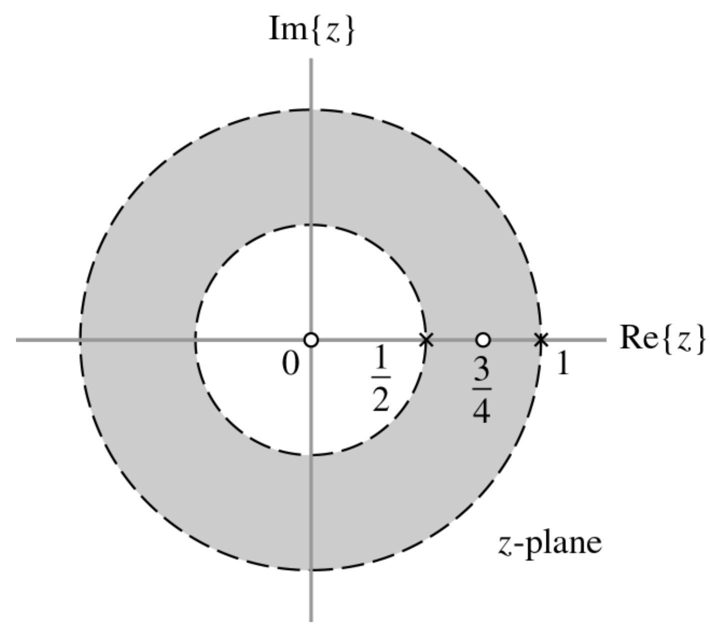
<Sol.>

1. Substituting for *x*[*n*] in Eq. (7.4), we obtain

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n} - u[-n-1] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n - \sum_{n=-\infty}^{-1} \left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n + 1 - \sum_{k=0}^{\infty} z^k.$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + 1 - \frac{1}{1-z}, \quad 1/2 < |z| < 1$$



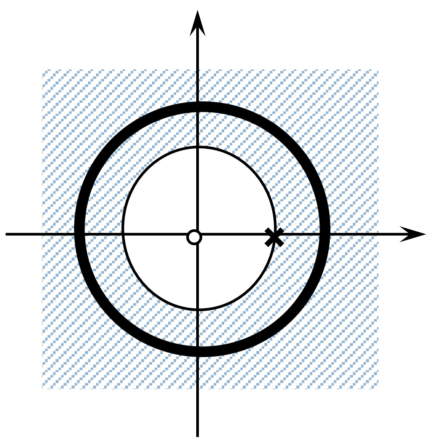
Properties of ROC

- Rational Function

$$X(z) = \frac{P(z)}{Q(z)}$$

- Ex.

$$x[n] = a^n u[n]$$



$$x[n] = -a^n u[-n-1]$$

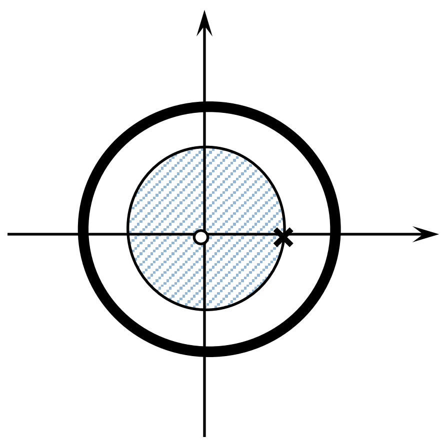


TABLE 4.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

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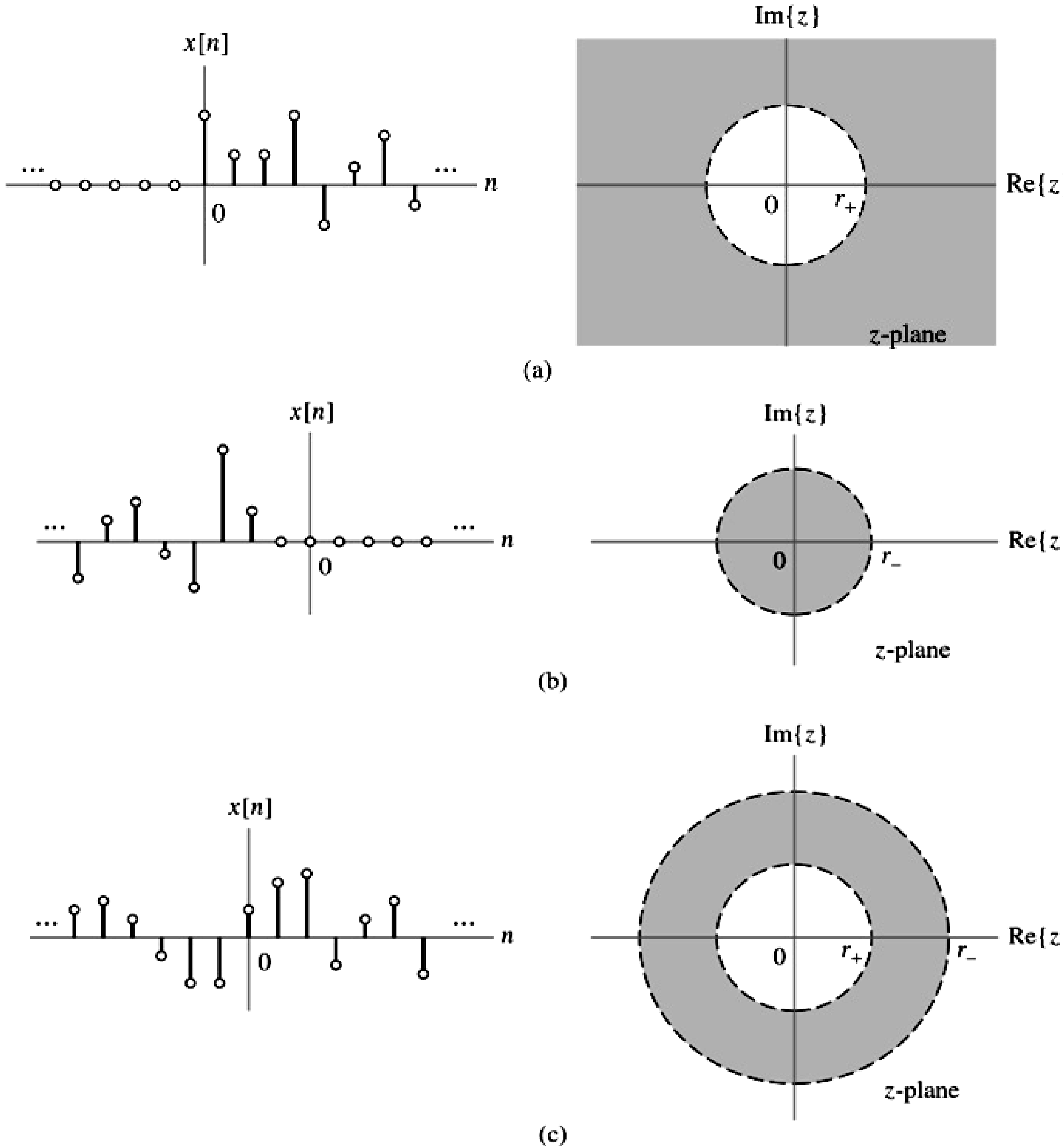
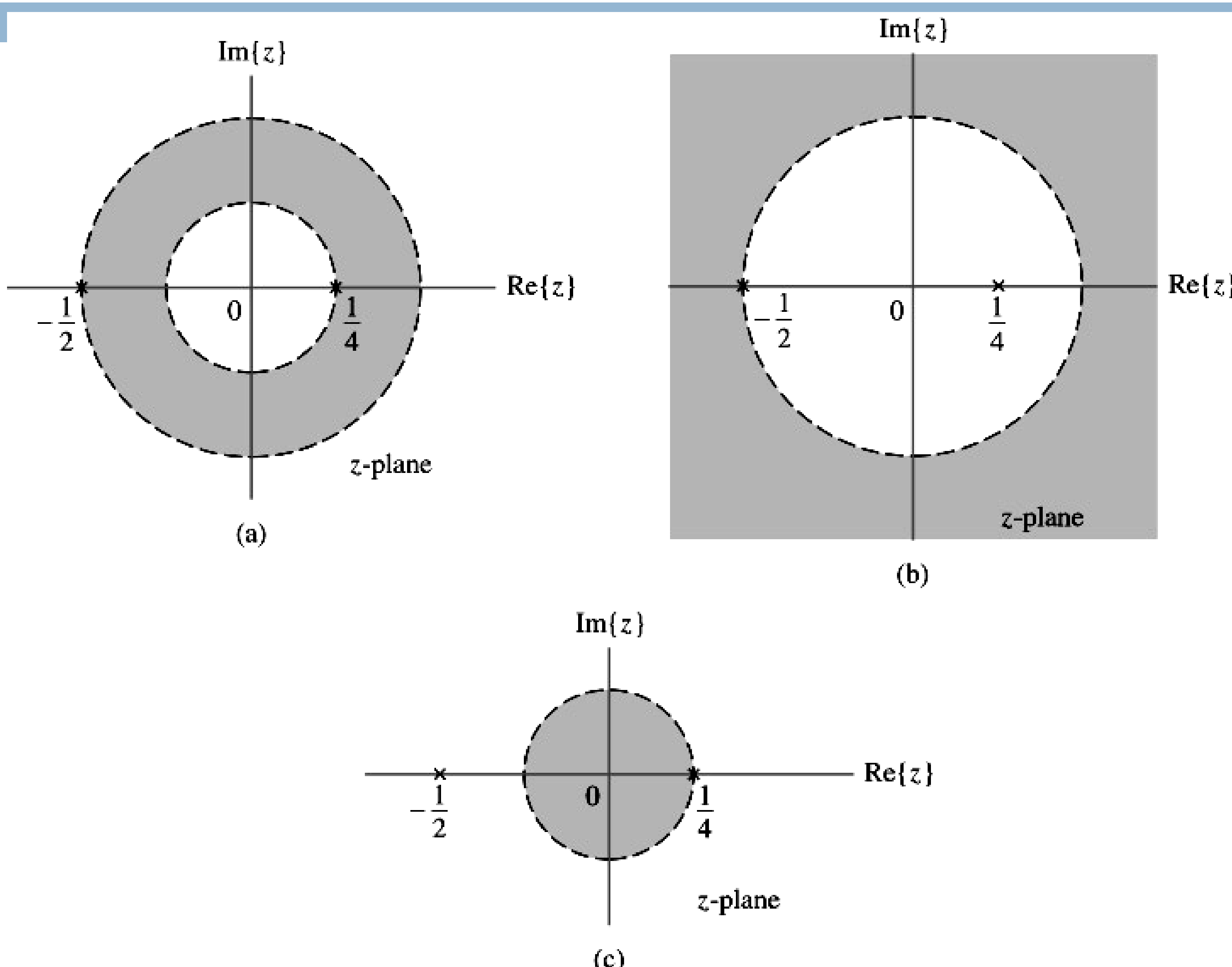
Properties of ROC

- Properties
  - The ROC is a ring or disk in the z-plane centered at the origin, i.e., $0 \le r_R < |z| < r_L \le \infty$
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
  - The ROC cannot contain any poles.
  - If x[n] is a finite-duration sequence, i.e. a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 \le \infty$, then the ROC is the entire z-plane except possibly z=0 or z= ∞.
  - If x[n] is a right-sided sequence, i.e. a sequence that is zero for $n<N_1< \infty$, the ROC extends outward from the outermost finite pole in X(z) to z= ∞ .
  - If x[n] is a left-sided sequence, i.e., a sequence that is zero for $n>N_2>-\infty$, the ROC extends inward from the innermost (smallest magnitude) nonzero pole in X(z) to (and possibly including) z=0.
  - A two-sided sequence is an infinite-duration sequence that is neither right-sided nor left-sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole, and, consistent with property 3, not containing any poles.
  - The ROC must be a connected region.

Properties of ROC



Properties of ROC



Properties Unilateral z Transform

1. Assume that

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with} \quad \text{ROC} \quad R_x$$

$$y[n] \overset{z}{\longleftrightarrow} Y(z), \quad \text{with} \quad \text{ROC} \quad R_y$$

♣ The ROC is changed by certain operations.

The ROC can be larger than the intersection if one or more terms in *x*[*n*] or *y*[*n*] cancel each other in the sum.

2. Linearity:

$$ax[n] + by[n] \overset{z}{\longleftrightarrow} aX(z) + bY(z), \quad \text{with} \quad \text{ROC} \quad \text{at} \quad \text{least} \quad R_x \cap R_y \quad \textbf{(7.11)}$$

♣ **Time Reversal**

$$x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right), \quad \text{with} \quad \text{ROC} \quad \frac{1}{R_x} \qquad \textbf{(7.12)}$$

Time reversal, or reflection, corresponds to replacing *z* by z^{-1}. Hence, if R_x is of the form $a < |z| < b$, the ROC of the reflected signal is $a < 1/|z| < b$, or $1/b < |z| < 1/a$.

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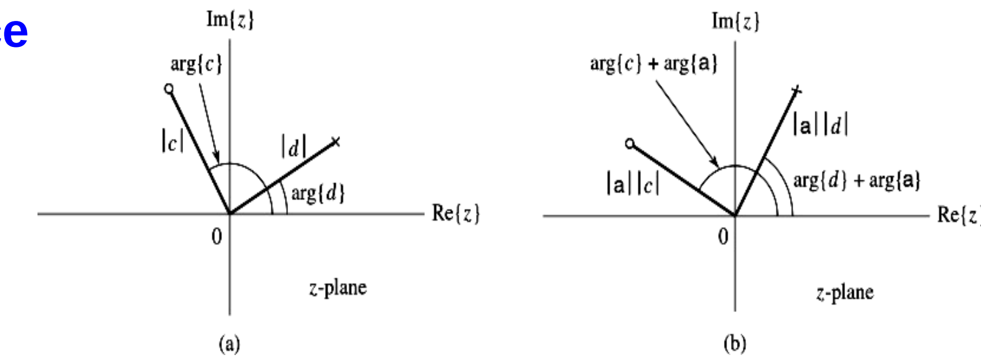
Properties Unilateral z Transform

♣ **Multiplication by an Exponential Sequence**



1. Let α be a complex number. Then

$$\alpha^n x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{\alpha}\right), \text{with} \quad \text{ROC} \quad |\alpha| R_x$$

2. The notation, $|\alpha| R_x$ implies that the ROC boundaries are multiplied by $|\alpha|$.

3. If R_x is $a < |z| < b$, then the new ROC is $|\alpha| a < |z| < |\alpha| b$.

4. If $X(z)$ contains a factor $1 - dz^{-1}$ in the denominator, so that d is pole, then $X(z/\alpha)$ has a factor $1 - \alpha dz^{-1}$ in the denominator and thus has a pole at αd.

5. If c is a zero of $X(z)$, then $X(z/\alpha)$ has a zero at αc.

6. This indicates that the poles and zeros of $X(z)$ have their radii changed by $|\alpha|$, and their angles are changed by $\arg\{\alpha\}$.

The Unilateral z-Transform

- Definition

$$X(z) \equiv Z[x[n]] \equiv \sum_{n=0}^{\infty} x[n] z^{-n}$$

- Time Delay

$$x[n] \quad \Longleftrightarrow \quad X(z)$$

$$x[n-k] \quad \Longleftrightarrow \quad z^{-k} X(z) + \sum_{n=1}^{k} x[-n] z^{-k+n}$$

$$x[n+k] \quad \Longleftrightarrow \quad z^{i} \left[X(z) - \sum_{n=0}^{k-1} x[n] z^{-n} \right]$$

Inversion of Unilateral z Transform

♣ **Convolution**

$$x[n]*y[n]\xleftrightarrow{z}X(z)Y(z),\quad \text{with}\quad \text{ROC}\quad \text{at least } R_x\cap R_y \qquad \textbf{(7.15)}$$

1. **Convolution of time-domain signals corresponds to multiplication of *z*-transforms.**
2. **The ROC may be larger than the intersection of R_x and R_y if a pole-zero cancellation occurs in the product *X*(*z*)*Y*(*z*).**

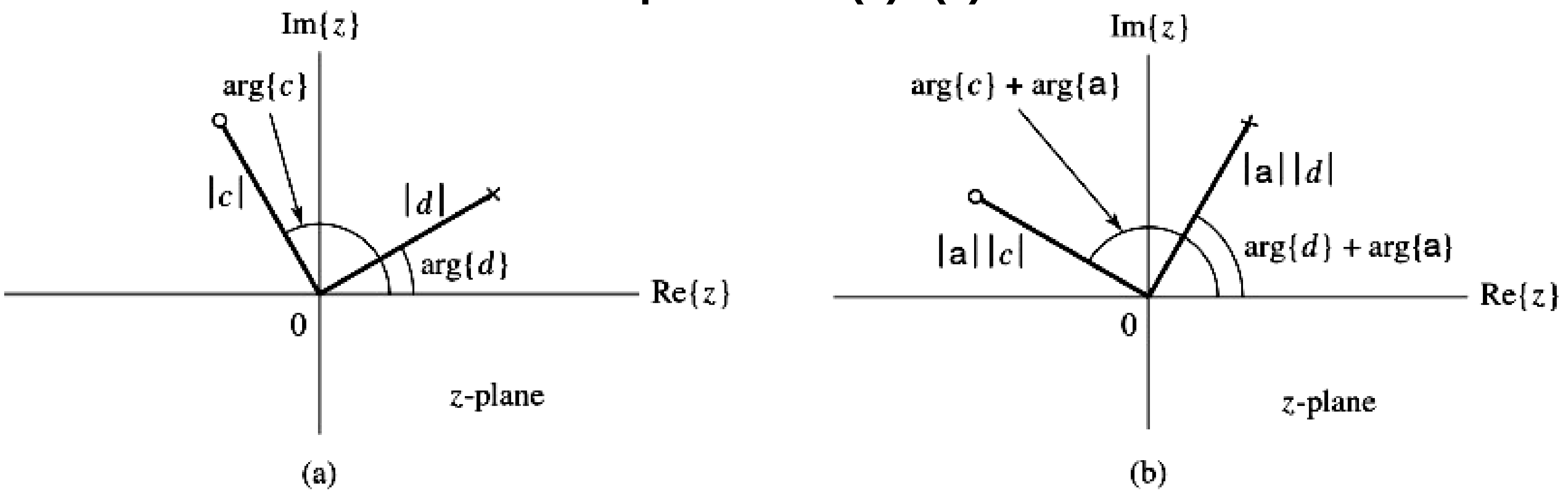


Figure 7.11 (p. 569)

The effect of multiplication by αn on the poles and zeros of a transfer function. (a) Locations of poles and zeros of *X*(*z*). (b) Locations of poles and zeros of *X*(*z*/α).

The Inverse z-Transform

- Methods
  - Direct Division
  - Partial Fraction Expansion

$$F(z) = \frac{-2z^4 + 5z^3 + z^2 - 6z + 3}{z^2 - z - 2}$$

- Direct Division
  - $F(z) = -2z^2 + 3z + 3z^{-2} + 3z^{-3} + 9z^{-4}$

$$
\begin{array}{r}
-2z^2 + 3z \qquad\qquad + 3z^{-2} + 3z^{-3} + 9z^{-4} \\
z^2 - z - 2 \overline{\big)\, -2z^4 + 5z^3 + z^2 - 6z + 3} \\
-2z^4 + 2z^3 + 4z^2 \qquad\qquad \\
\hline
3z^3 - 3z^2 - 6z + 3 \\
3z^3 - 3z^2 - 6z \quad \\
\hline
+3 \quad
\end{array}
$$

$f[k] = \{-2, 3, 0, 0, 3, 3, 9, \ldots\}$

↑ (under the first 0)

- Ex. $3/(z^2 - z - 2)$

Inverse z-transform by Power Series Expansion

Example 3.11 Inverse Transform by Power Series Expansion

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

Therefore,

$$x[n] = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$

The Inverse z-Transform (c.1)

- Partial Fraction Expansion and Table Lookup

$$X(z)=\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

  - If M<N and the poles are all first order

$$X(z)=\frac{b_0}{a_0}\sum_{k=1}^{N}\frac{A_k}{(1-d_k z^{-1})} \qquad A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

  - If M>= N and the poles are all first order, the complete partial fraction expression can be

$$X(z)=\sum_{r=0}^{M-N}B_r z^{-r}+\sum_{k=1}^{N}\frac{A_k}{(1-d_k z^{-1})} \qquad A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

  - If X(z) has multiple-order poles and M>=N

$$X(z)=\sum_{r=0}^{M-N}B_r z^{-r}+\sum_{k=1,k\neq i}^{N}\frac{A_k}{(1-d_k z^{-1})}+\sum_{m=1}^{s}\frac{C_m}{(1-d_i z^{-1})^m}$$

$$C_m=\frac{1}{(s-m)!(-d_i)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}[(1-d_i w)^s X(w^{-1})]\right\}_{w=d_i^{-1}}$$

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Transfer Function

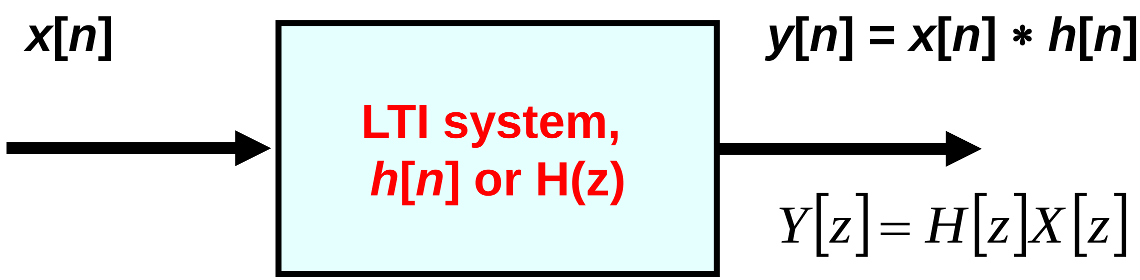
1. Output of LTI system:

$$y[n] = h[n] * x[n] \quad \xrightarrow{\text{Taking } z\text{-transform}} \quad Y[z] = H[z]X[z] \qquad \textbf{(7.19)}$$

2. Transfer function:

$$H[z] = \frac{Y[z]}{X[z]} \qquad \textbf{(7.20)}$$

♣ **This definition applies at all *z* in the ROC of *X*(*z*) and *Y*(*z*) for which *X*(*z*) is nonzero.**



Transfer Function

Example 7.13 ***System Identification***

The problem of finding the system description from knowledge of input and output is know as system identification. Find the transfer function and impulse response of a causal LTI system if the input to the system is

$$x[n] = (-1/3)^n u[n]$$

and the output is

$$y[n] = 3(-1)^n u[n] + (1/3)^n u[n]$$

<Sol.> **1. The *z*-transforms of the input and output are respectively given by**

$$X(z) = \frac{1}{1+(1/3)z^{-1}} \quad \text{with ROC } |z| > 1/3$$

and

$$Y(z) = \frac{3}{1+z^{-1}} + \frac{1}{1-(1/3)z^{-1}}$$

$$= \frac{4}{(1+z^{-1})(1-(1/3)z^{-1})}, \quad \text{with ROC } |z| > 1$$

Transfer Function

2. We apply Eq.(7.20) the obtain the transfer function:

$$H(z) = \frac{4(1-(1/3)z^{-1})}{(1+z^{-1})(1-(1/3)z^{-1})}, \quad \text{with} \quad \text{ROC} \quad |z| > 1$$

3. The impulse response of the system is obtain by finding the inverse *z*-transform of *H*(*z*). Applying a partial fraction expansion to *H*(*z*) yields

$$H(z) = \frac{2}{1+z^{-1}} + \frac{2}{1-(1/3)z^{-1}}, \quad \text{with} \quad \text{ROC} \quad |z| > 1$$

4. The impulse response is thus given by

$$h[n] = 2(-1)^n u[n] + 2(1/3)^n u[n]$$

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Solving Difference Equations with Initial Conditions

♣ **Differentiation in the z-Domain**

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{d}{dz}X(z), \quad \text{with} \quad \text{ROC } R_x \qquad \textbf{(7.16)}$$

1. **Multiplication by *n* in the time domain corresponds to differentiation with respect to *z* and multiplication of the result by – *z* in the *z*-domain.**
2. **This operation does not change the ROC.**

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Transfer Function

7.6.1 *Relating the Transfer Function and the Difference equation*

1. *N*th-order difference equation:

$$\sum_{K=0}^{\infty} a_k y[n-k] = \sum_{K=0}^{M} b_k x[n-k]$$

2. The transfer function *H*(*z*) is an eigenvalue of the system associated with the eigenfunction z^n.

If $x[n] = z^n$, then the output of an LTI system is $y[n] = z^n H(z)$. Substituting $x[n-k] = z^{n-k} H(z)$ into the difference equation gives the relationship

$$z^n \sum_{K=0}^{N} a_k z^{-k} H(z) = z^n \sum_{K=0}^{M} b_k z^{-k}$$

3. Transfer function:

Rational transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (7.21)$$

See page 581, textbook.

Transfer Function

Example 7.14 ***Finding the Transfer Function and Impulse Response***

Determine the transfer function and the impulse response for the causal LTI system described by the difference equation

$$y[n]-(1/4)y[n-1]-(3/8)y[n-1]=-x[n]+2x[n-1]$$

<Sol.>

1. We obtain the transfer function by applying Eq.(7.21):

$$H(z)=\frac{-1+2z^{-1}}{1-(1/4)z^{-1}-(3/8)z^{-2}}$$

2. The impulse response is found by identifying the inverse *z*-transform of *H*(*z*). Applying a partial-fraction expansion to *H*(*z*) give

$$H(z)=\frac{-2}{1+(1/2)z^{-1}}+\frac{1}{1-(3/4)z^{-1}}$$

3. The system is causal, so we choose the right-side inverse *z*-transform for each term to obtain the following impulse response:

$$h[n]-2(-1/2)^{n}u[n]+(3/4)^{n}u[n]$$

Transfer Function

Example 7.15 ***Finding a Difference-Equation Description***

Find the difference-equation description of an LTI system with transfer function

$$H(z)=\frac{5z^{-1}+2z^{-2}}{(1+3z^{-1}+2z^{-2})}$$

<Sol.>

1. We rewrite $H(z)$ as a ratio of polynomials in z^{-1}. Dividing both the numerator and denominator, we obtain

$$H(z)=\frac{5z^{-1}+2z^{-2}}{(1+3z^{-1}+2z^{-2})}$$

2. Comparing transfer function with Eq.(7.21), we conclude that M = 2, N = 2, b_0 = 0, b_1 = 5, b_2 = 2, a_0 = 1, a_1 = 3, and a_2 = 2. Hence, this system is described by the difference equation

$$y[n]+3y[n-1]+2y[n-2]=5x[n-1]+2x[n-2]$$

♣ **Transfer function in pole-zero form:**

where $c_k \equiv$ zeros; $d_k \equiv$ poles; and

$$\tilde{b}=b_0/a_0 \equiv \text{gain factor}$$

$$H(z)=\frac{\tilde{b}\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \quad \textbf{(7.22)}$$

Causality and Stability

2. The impulse response of a stable LTI system is absolutely summable and the DTFT of the impulse response exists.

⟹ The ROC must includes the unit circle in the *z*-plane.

Pole d_k inside the unit circle, i.e., $|d_k| < 1$

Left-sided inverse transform ⟶ Exponentially decaying term

Pole d_k outside the unit circle, i.e., $|d_k| > 1$

Right-sided inverse transform ⟶ Exponentially increasing term

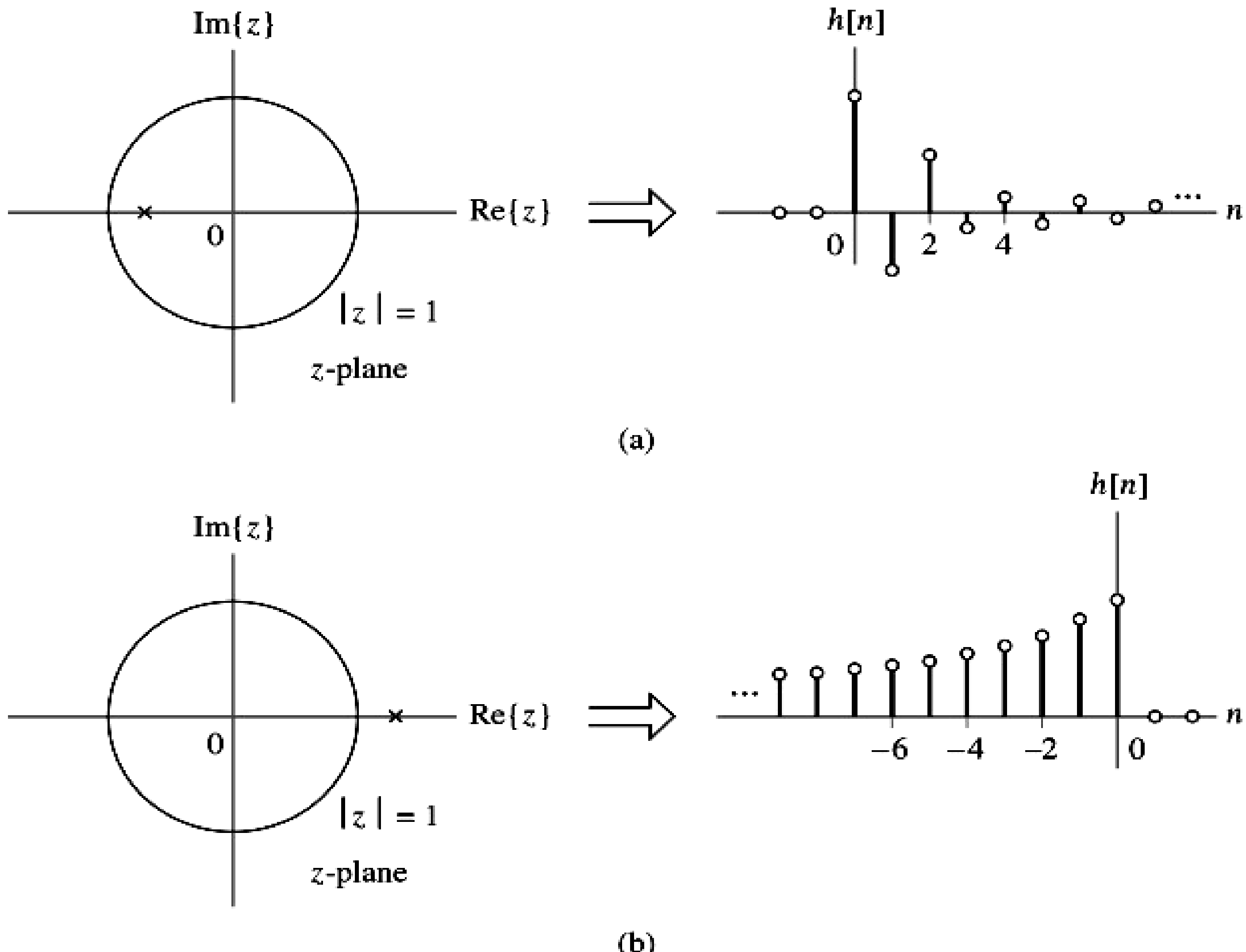
⟹ Fig. 7.15

- Note that a stable impulse response cannot contain any increasing exponential or sinusoidal terms, since then the impulse response is not absolutely summable.
- LTI system are both stable and causal must have all their poles inside the unit circle.

⟹ Fig. 7.16

Causality and Stability

Figure 7.15 (p. 583) The relationship between the location of a pole and the impulse response characteristics for a stable system. (a) A pole inside the unit circle contributes a right-sided term to the impulse response. (b) A pole outside the unit circle contributes a left-sided term to the impulse response.



Causality and Stability

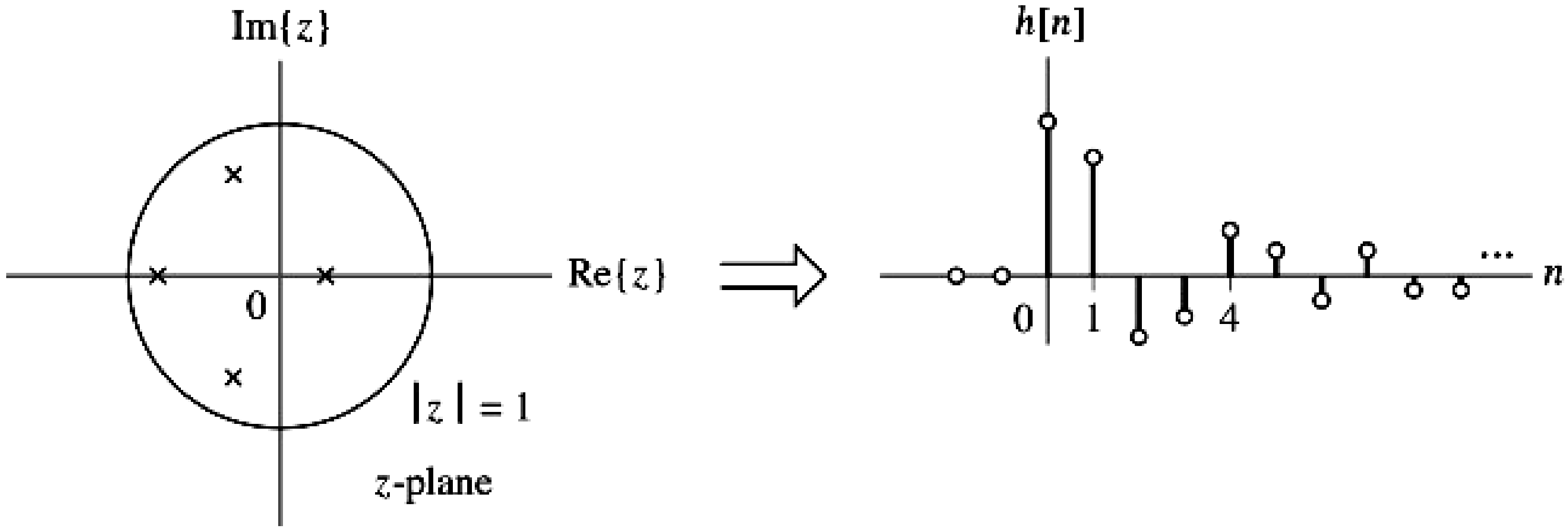


Figure 7.16 (p. 584)
A system that is both stable and causal must have all its poles inside the unit circle in the *z*-plane, as illustrated here.

Causality and Stability

7.7.1 *Inverse System*

1. The impulse response of an inverse system, $h^{inv}[n]$, satisfies

$$h^{inv}[n] * h[n] = \delta[n]$$

where $h[n]$ is the impulse response of the system to be inverted.

2. Taking z-transform:

$$H^{inv}(z)H(z) = 1 \quad \Rightarrow \quad H^{inv}(z) = \frac{1}{H(z)}$$

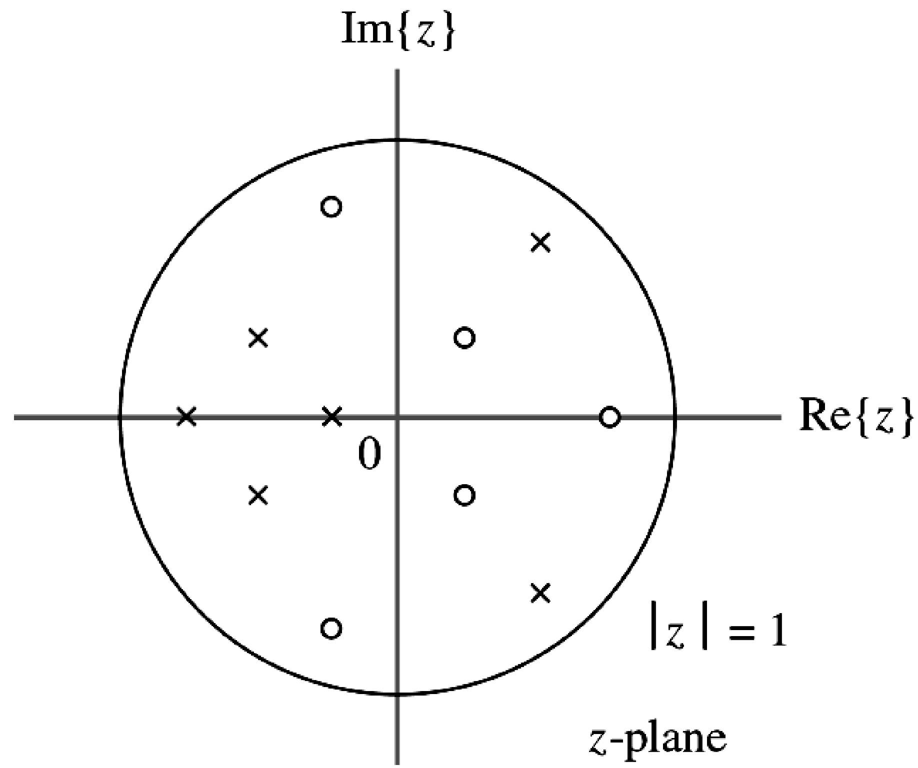
3. If $H(z)$ is written in pole-zero form shown in Eq. (7.23), then

$$H^{inv}(z) = \frac{z^{-1}\prod_{k=1}^{N-l}(1-d_k z^{-1})}{\tilde{b} z^{-p}\prod_{k=1}^{M-p}(1-c_k z^{-1})} \qquad \textbf{(7.24)}$$

♣ Any system described by a rational transfer function has an inverse system of this form.

Causality and Stability

Figure 7.18 (p. 586)
A system that has a causal and stable inverse must have all its poles and zeros inside the unit circle, as illustrated here.



4. **$H^{inv}(z)$ is both stable and causal if all of its poles are inside the unit circle. Since the poles of $H^{inv}(z)$ are the zeros of $H(z)$, we conclude that a stable and causal inverse of an LTI system $H(z)$ exists if and only if all the zeros of $H(z)$ are inside the unit circle.**

5. **A system with all its poles and zeros inside the unit circle, as illustrated in Fig. 7.18, is termed a *minimum-phase* system.**

♣ **The phase response of a minimum-phase system is uniquely determined by the magnitude response, and vice versa.**

Causality and Stability

Example 7.18 ***Stable and Causal Inverse System***

An LTI system is described by the difference equation

$$y[n]-y[n-1]+\frac{1}{4}y[n-2]=x[n]+\frac{1}{4}x[n-1]+\frac{1}{8}x[n-2]$$

Find the transfer function of the inverse system. Does a stable and causal LTI inverse system exist?

<Sol.> **1. We find the transfer function of the given system by applying Eq.(7.21) to obtain**

$$H(z)=\frac{1+\frac{1}{2}z^{-1}-\frac{1}{8}z^{-2}}{1-z^{-1}+\frac{1}{4}z^{-2}}$$

$$=\frac{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})^{2}}$$

2. The inverse system then has the transfer function

$$H^{inv}(z)=\frac{(1-\frac{1}{2}z^{-1})^{2}}{(1+\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})}$$

PSPLAB

Determining the Frequency Response from Poles and Zeros

- Impulse Response for Rational Functions

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \longleftrightarrow h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

  - Infinite Impulse Reponse (IIR) Systems
    - The length of the impulse response is infinite.
  - Finite Impulse Response (FIR) Systems
    - The length of the impulse reponse is finite.

How to check ?

- Examples

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]$$

$$y[n] - ay[n-1] = x[n] - a^{M+1} x[n-M-1]$$

PSPLAB

Determining the Frequency Response from Poles and Zeros

- A stable linear time-invariant system
  - Rational Function

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

  - Magnitude Response

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1 - d_k e^{-j\omega}|} \qquad |H(e^{j\omega})|^2 = \left|\frac{b_0}{a_0}\right|^2 \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})(1 - c^*_k e^{j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})(1 - d^*_k e^{j\omega})}$$

  - Gain (dB)

$$20\log_{10}|H(e^{j\omega})| = 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}|1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20\log_{10}|1 - d_k e^{-j\omega}|$$

    - $|H(e^{j\omega})| = 2^m$ is approximately 6m dB, while $|H(e^{j\omega})| = 10^m$ is approximately 20m dB

$$20\log_{10}|Y(e^{j\omega})| = 20\log_{10}|H(e^{j\omega})| + 20\log_{10}|X(e^{j\omega})|$$

PSPLAB

Determining the Frequency Response from Poles and Zeros

- Phase Response (c.1)

$$\angle H(e^{j\omega}) =$$

$$\angle\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} \angle[1 - c_k e^{-j\omega}) - \sum_{k=1}^{N} \angle[1 - d_k e^{-j\omega})$$

  - The principal value of the phase is denoted as ARG[H(e^{jw})]

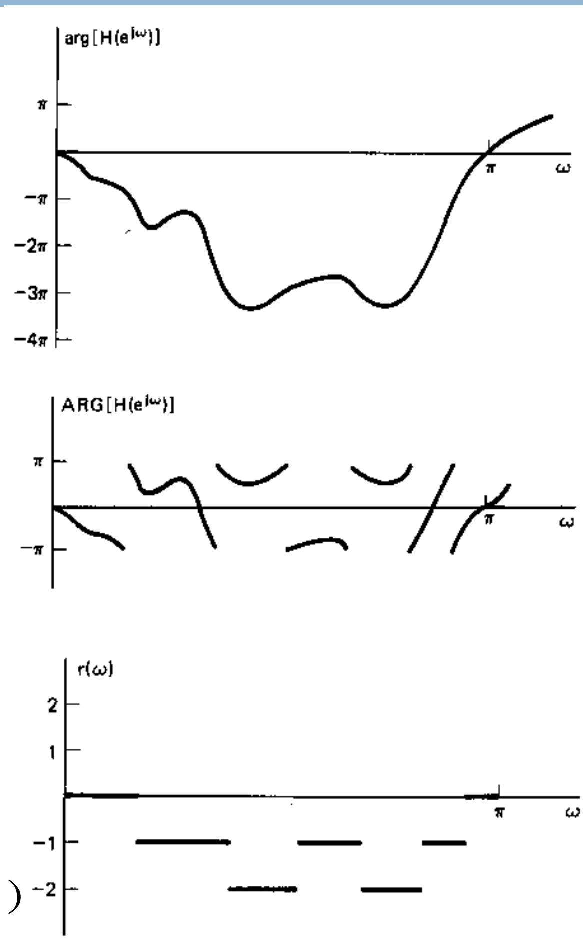
$$-\pi < ARG[H(e^{j\omega})] \le \pi$$

$$\angle H(e^{j\omega}) = ARG[H(e^{j\omega})] + 2\pi r(\omega)$$

  - Principal Values = Sum of Individual PVs

$$ARG[H(e^{j\omega})] = ARG\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M} ARG[1 - c_k e^{-j\omega})$$

$$- \sum_{k=1}^{N} ARG[1 - d_k e^{-j\omega}) + 2\pi r(\omega)$$



Determining the Frequency Response from Poles and Zeros

- Phase Distortion and Delay
  - Observation 1

Delay — Linear Phase

$$h_{id}[n] = \delta[n - n_d] \Longleftrightarrow H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

  - The Ideal Lowpass Filter with linear phase

Ideal Filters with Causality ?

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \Longleftrightarrow h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \quad -\infty < n < \infty$$

  - Observation 2-- A narrow band signal s[n]cos(ω_0n)
    - The phase for the ω_0 can be approximated as $\angle H(e^{j\omega}) \approx -\phi_0 - \omega n_d$

$$y[n] = s[n - n_d]\cos(\omega_0 n - \phi_0 - \omega_0 n_d)$$

  - Group Delay-- A measure for the nonlinearity of the phase

$$\tau(\omega) = grd[H(e^{j\omega})] = -\frac{d}{d\omega}\left\{\angle H(e^{j\omega})\right\}$$

PSPLAB

Determining the Frequency Response from Poles and Zeros

- Phase Response
  - Alternative relation

$$ARG[H(e^{j\omega})] = \arctan\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right]$$

$$\frac{d(\arctan x)}{dx} = \lim_{\Delta x \to 0} \frac{\arctan(x+\Delta x) - \arctan x}{\Delta x} = \frac{1}{1+x^2}.$$

- Group Delay
  - Derivative of the continuous phase function

$$grd[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \sum_{k=1}^{N}\frac{d}{d\omega}(\arg[1-d_k e^{-j\omega}] - \sum_{k=1}^{M}\frac{d}{d\omega}(\arg[1-c_k e^{-j\omega}])$$

  - That is

$$grd[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{|d_k|^2 - \mathrm{Re}\{d_k e^{-j\omega}\}}{1+|d_k|^2 - 2\mathrm{Re}\{d_k e^{-j\omega}\}} - \sum_{k=1}^{M}\frac{|c_k|^2 - \mathrm{Re}\{c_k e^{-j\omega}\}}{1+|c_k|^2 - 2\mathrm{Re}\{c_k e^{-j\omega}\}}$$

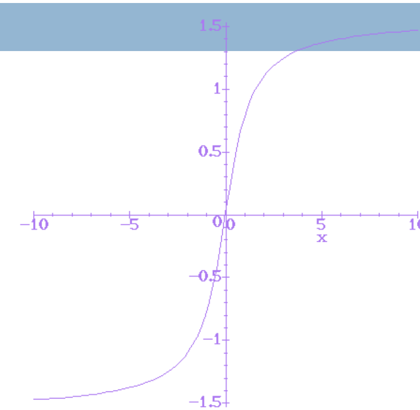
  - Can be obtained from the principle values except at discontinuities.

PSPLAB

Determining the Frequency Response from Poles and Zeros

- Single Pole or Zero
  - The form

$$(1 - pz^{-1})$$



  - The magnitude squared

$$\left|1 - re^{j\theta}e^{-j\omega}\right|^2 = (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega}) = 1 + r^2 - 2r\cos(\omega - \theta)$$

  - The log magnitude in dB is

$$20\log_{10}\left|1 - re^{j\theta}e^{-j\omega}\right| = 10\log_{10}[1 + r^2 - 2r\cos(\omega - \theta)]$$

  - The phase

$$ARG\left[1 - re^{j\theta}e^{-j\omega}\right] = \arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$$

  - Group Delay

$$grd\left[1 - re^{j\theta}e^{-j\omega}\right] = \frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{r^2 - r\cos(\omega - \theta)}{\left|1 - re^{j\theta}e^{-j\omega}\right|^2}$$

Determining the Frequency Response from Poles and Zeros

- Single Pole or Zero
  - Group Delay

$$ARG\left[1-re^{j\theta}e^{-j\omega}\right]=\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

$$grd\left[1-re^{j\theta}e^{-j\omega}\right]=-\frac{1}{1+\left(\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right)^2}\left(\frac{r\cos(\omega-\theta)}{1-r\cos(\omega-\theta)}+\frac{-r^2\sin^2(\omega-\theta)}{\left(1-r\cos(\omega-\theta)\right)^2}\right)$$

$$=-\frac{\left(1-r\cos(\omega-\theta)\right)^2}{1+r^2-2r\cos(\omega-\theta)}\left(\frac{r\cos(\omega-\theta)\left(1-r\cos(\omega-\theta)\right)}{\left(1-r\cos(\omega-\theta)\right)^2}+\frac{-r^2\sin^2(\omega-\theta)}{\left(1-r\cos(\omega-\theta)\right)^2}\right)$$

$$=\frac{r^2-r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}$$

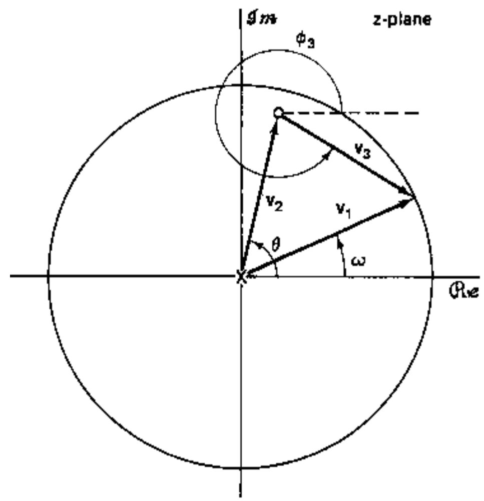
Determining the Frequency Response from Poles and Zeros

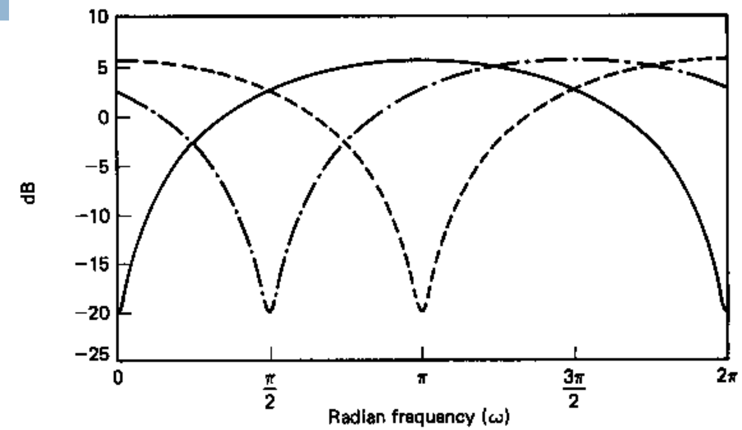
- Ex.

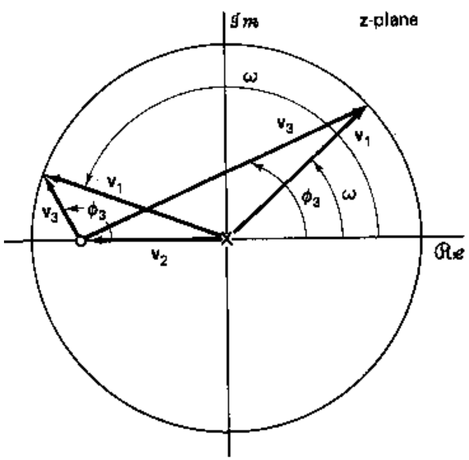
$$(1 - pz^{-1})$$

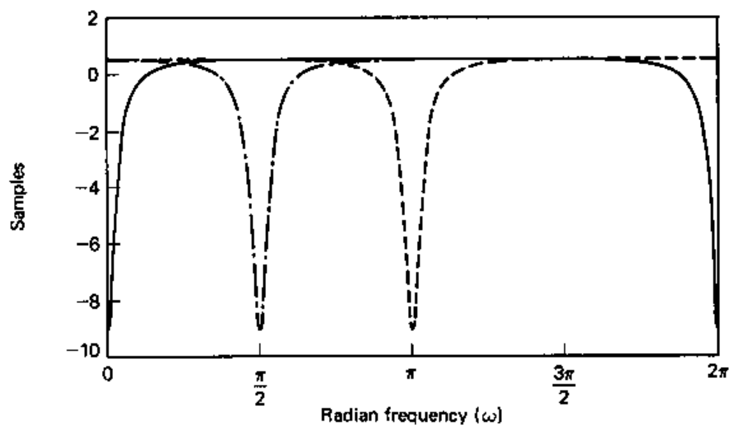
$$\frac{|v_3|}{|v_1|}$$

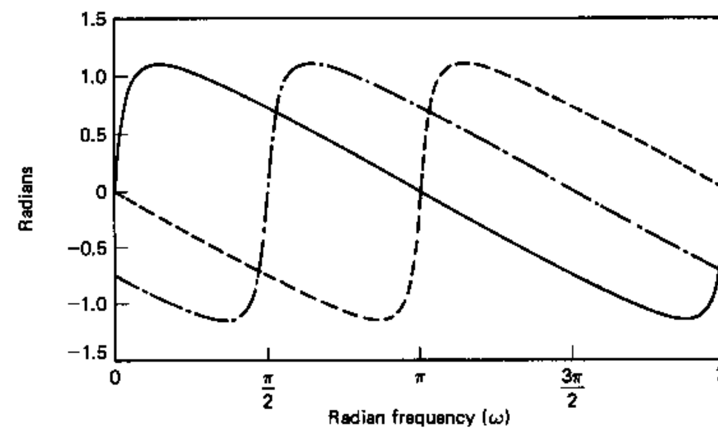
$$\phi_3 - \phi_1 = \phi_3 - \omega$$





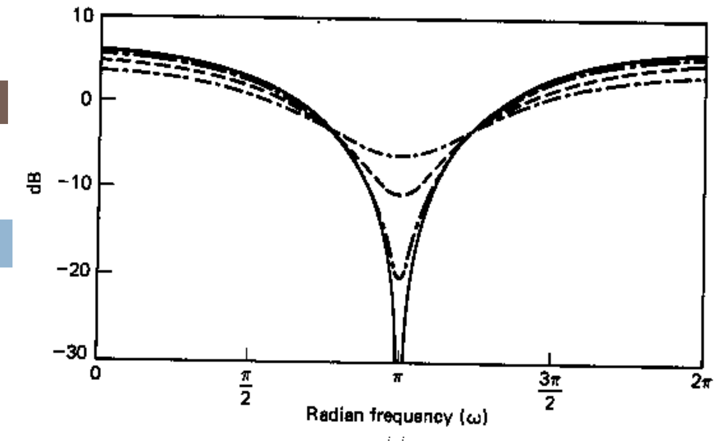


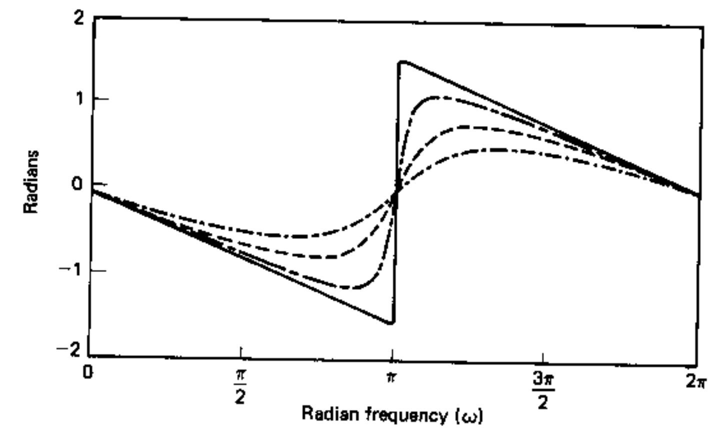


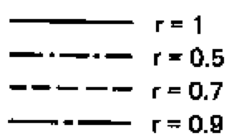


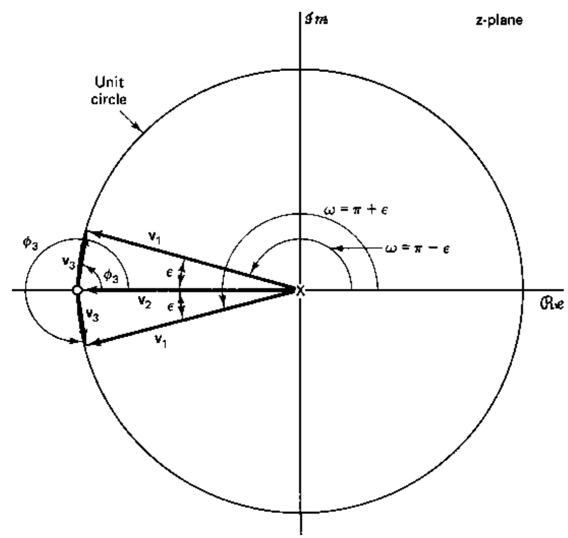
Determining the Frequency Response from Poles and Zeros

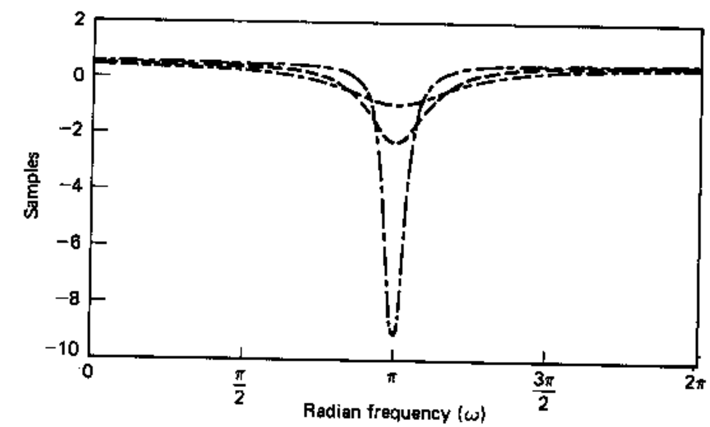
- Frequency Responsee for a Single Zero at π





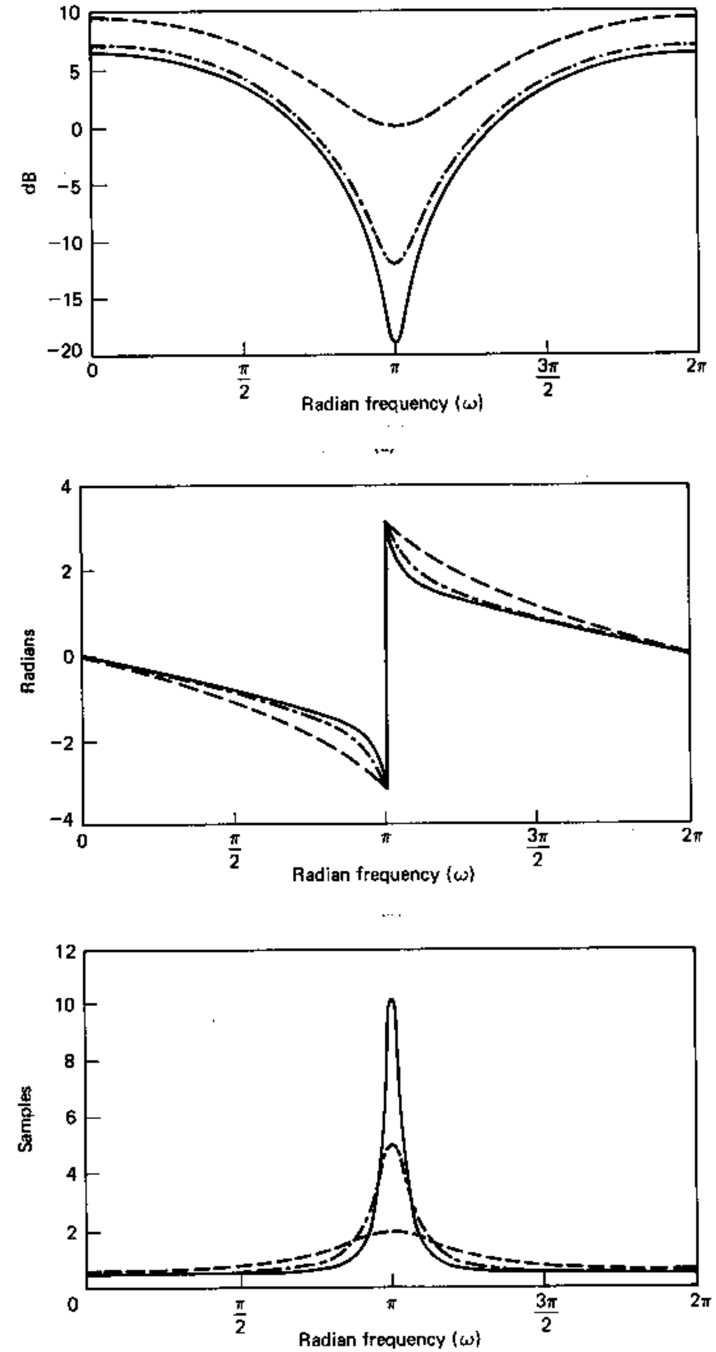




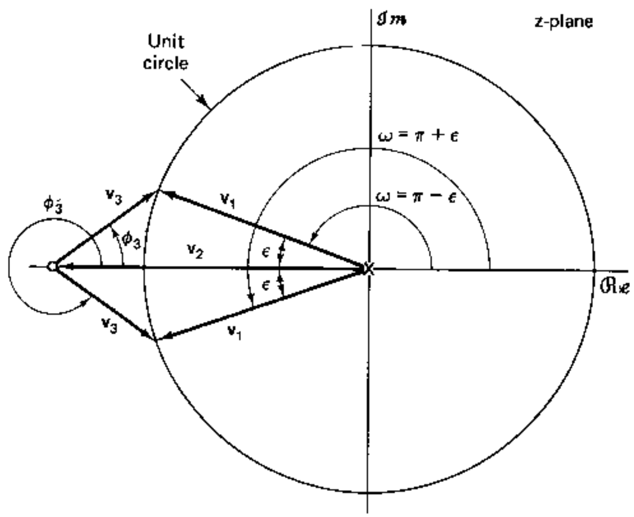


Determining the Frequency Response from Poles and Zeros

- Frequency Response for a Single Zero near π



r = 1/0.9
r = 1.25
r = 2.0



Determining the Frequency Response from Poles and Zeros

- **Relationship between the locations of poles and zeros in the *z*-plane and the frequency response of the system:**

1. Transfer function:

Substituting $e^{j\Omega}$ for z in $H(z)$

2. Assume that the ROC includes the unit circle. Substituting $z = e^{j\Omega}$ into Eq. (7.23) gives

$$H(e^{j\Omega}) = \frac{\tilde{b}\, e^{-jp\Omega} \prod_{k=1}^{M-p} (1 - c_k e^{-j\Omega})}{e^{-jl\Omega} \prod_{k=1}^{N-l} (1 - d_k e^{-j\Omega})}$$

$$H(e^{j\Omega}) = \frac{\tilde{b}\, e^{-j(N-M)\Omega} \prod_{k=1}^{M-p} (e^{-j\Omega} - c_k)}{\prod_{k=1}^{N-l} (e^{-j\Omega} - d_k)} \qquad (7.25)$$

3. The magnitude of $H(e^{j\Omega})$ at some fixed value of Ω, say, Ω_0, is defined by

$$H(e^{j\Omega}) = \frac{\left|\tilde{b}\right| \prod_{k=1}^{M-p} \left|(e^{j\Omega_o} - c_k)\right|}{\prod_{k=1}^{N-l} (e^{j\Omega_0} - d_k)}$$

- **$|e^{j\Omega_0} - g| \equiv$ product term, where g represents either a pole or a zero.**

Determining the Frequency Response from Poles and Zeros

4. Vector representation of $e^{j\Omega} - g$: Fig. 7. 19.

1) $e^{j\Omega_0} \equiv$ a vector from the origin to $e^{j\Omega_0}$; $g \equiv$ a vector from the origin to g.
2) We assess the contribution of each pole and zero to the overall frequency response by examining $|e^{j\Omega_0} - g|$ as Ω_0 changes.

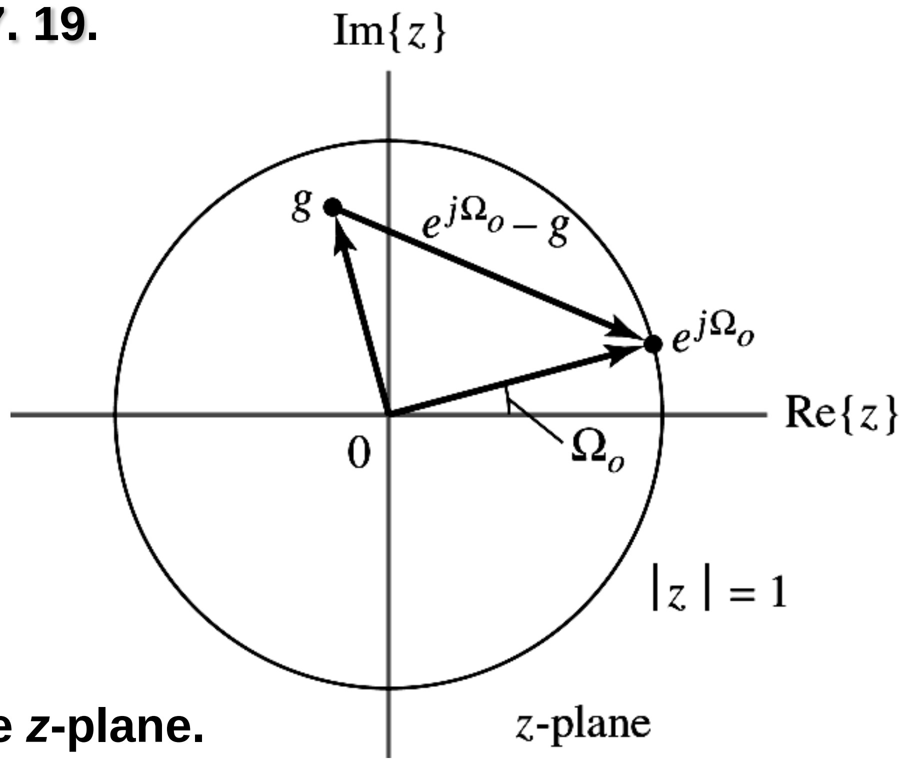


Figure 7.19 (p. 589)
Vector interpretation of $e^{j\Omega_0} - g$ in the z-plane.

Determining the Frequency Response from Poles and Zeros

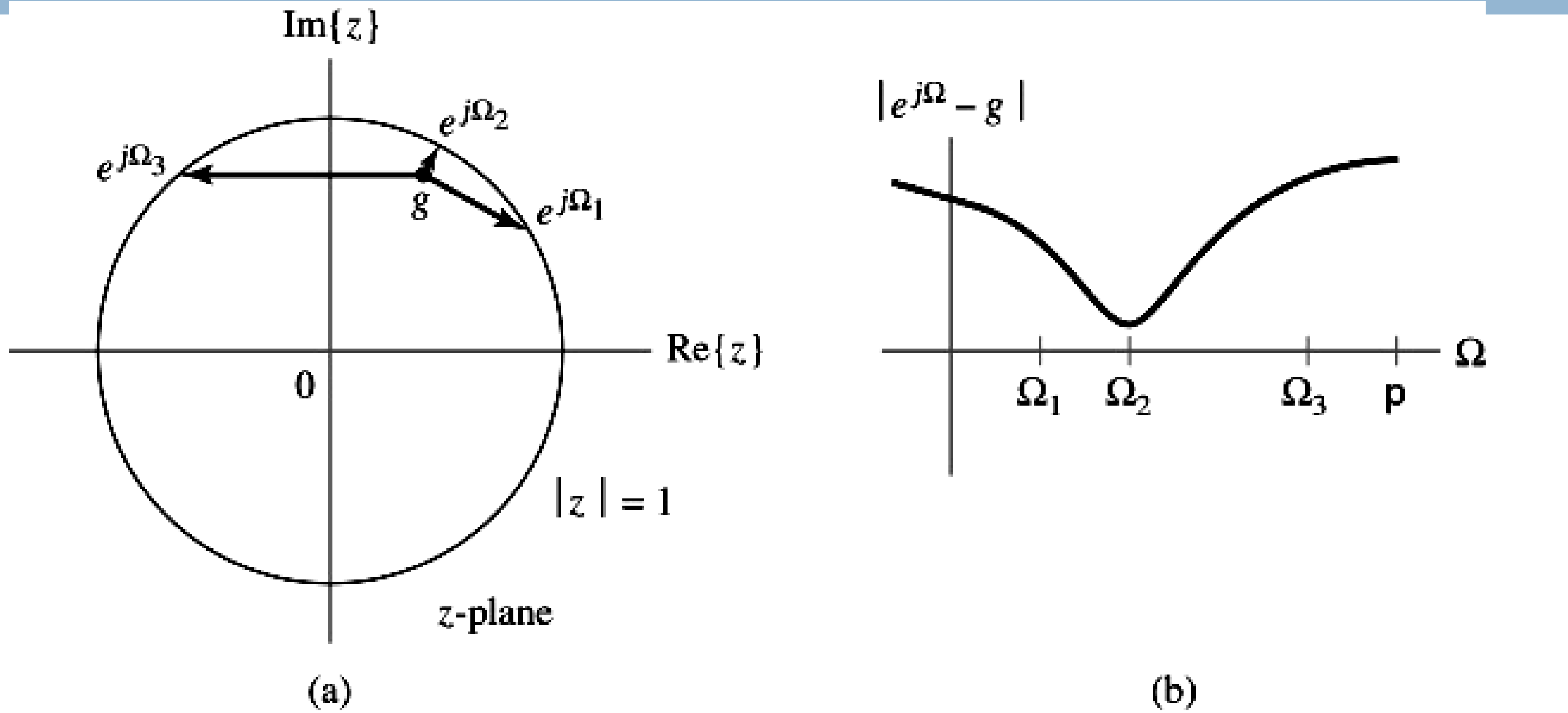


Figure 7.20 (p. 589)
The quantity $|e^{j\Omega} - g|$ is the length of a vector from g to $e^{j\Omega}$ in the z-plane. (a) Vectors from g to $e^{j\Omega}$ at several frequencies. (b) The function $e^{j\Omega} - g|$.

5. Fig. 7.20(a) depicts the vector $|e^{j\Omega_0} - g|$ for several different values of Ω; while Fig. 7.20(b) depicts $|e^{j\Omega_0} - g|$ as a continuous function of frequency.

Determining the Frequency Response from Poles and Zeros

- If $\Omega = \arg\{g\}$, then $|e^{j\Omega} - g|$ attains its minimum value of $1 - |g|$ when g is inside the unit circle and takes on the value $|g| - 1$ when g is outside the unit circle. Hence, if g is close to the unit circle ($|g| \approx 1$), then $|e^{j\Omega} - g|$ becomes very small when $\Omega = \arg\{g\}$.

6. If g represents a zero, then $|e^{j\Omega} - g|$ contributes to the numerator of $|H(e^{j\Omega})|$. Thus, at frequencies near $\arg\{g\}$, $|H(e^{j\Omega})|$ tends to have a minimum.
7. If g represents a pole, then $|e^{j\Omega} - g|$ contributes to the denominator of $|H(e^{j\Omega})|$. When $|e^{j\Omega} - g|$ decreases, $|H(e^{j\Omega})|$ increases, with the size of the increase dependent on how far the pole is from the unit circle.

Determining the Frequency Response from Poles and Zeros

Example 7.21 ***Magnitude Response from Poles and Zeros***

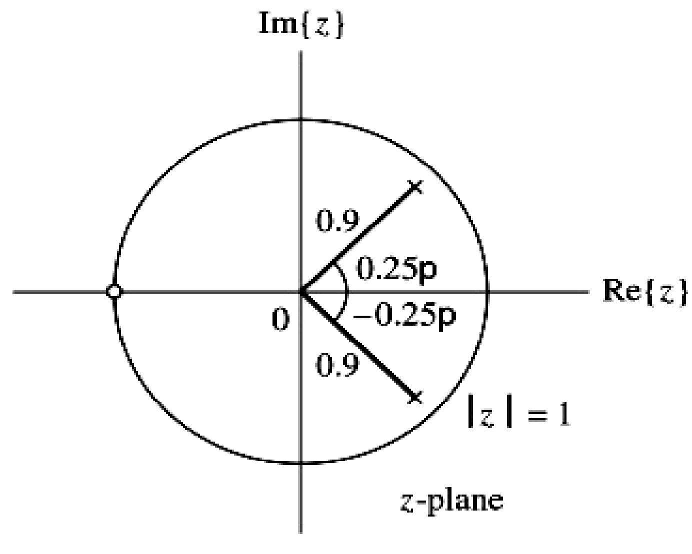
Sketch the magnitude response for an LTI system having the transfer function

$$H(z) = \frac{1+z^{-1}}{(1-0.9e^{j\frac{\pi}{4}}z^{-1})(1-0.9e^{-j\frac{\pi}{4}}z^{-1})}$$

<Sol.>

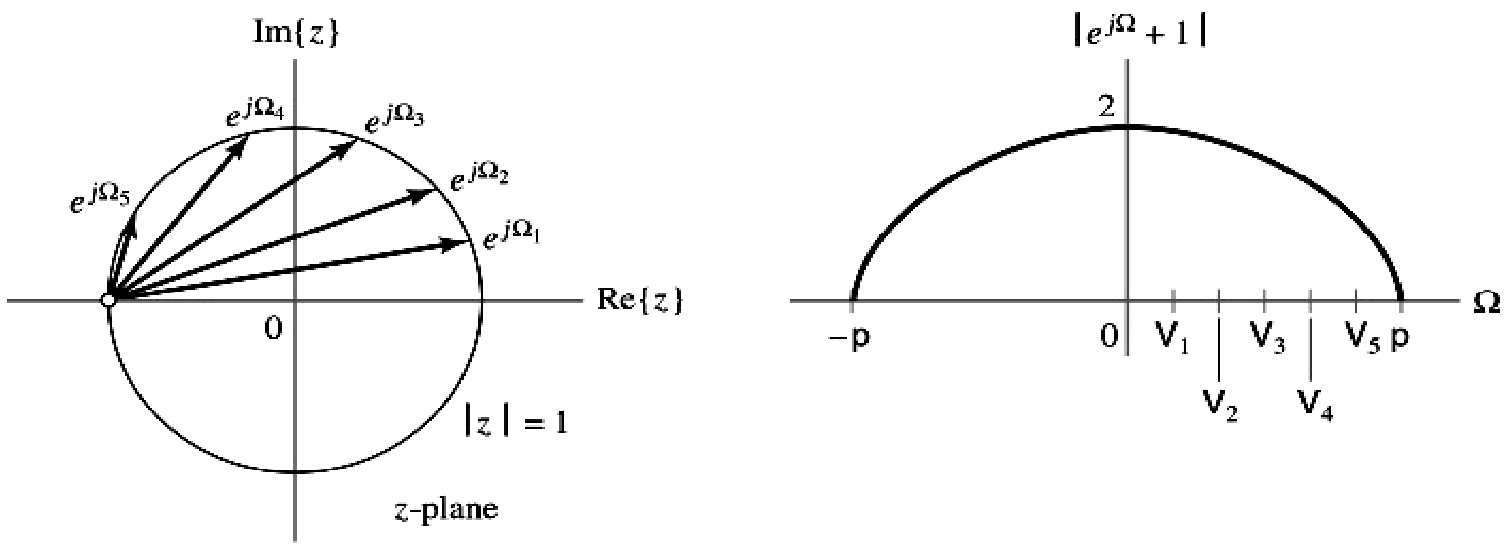
The system has a zero at $z = -1$ and poles at $z = 0.9e^{j\pi/4}$ and $z = 0.9e^{-j\pi/4}$ as depicted in Fig.7.23(a). Hence, the magnitude response will be zero at $\Omega = \pi$ and will be large at $\Omega = \pm\pi/4$, because the poles are close to the unit circle. Figures 7.23 (b)-(d) depict the component of the magnitude response associated with the zero and each poles. Multiplication of these contributions gives the overall magnitude response sketched in Fig. 7.23(e).

Figure 7.23a (p. 592)
Solution for Example 7.21.
(a) Locations of poles and zeros in the *z*-plane.
(b) The component of the magnitude response associated with a zero is given by the length of a vector from the zero to $e^{j\Omega}$.



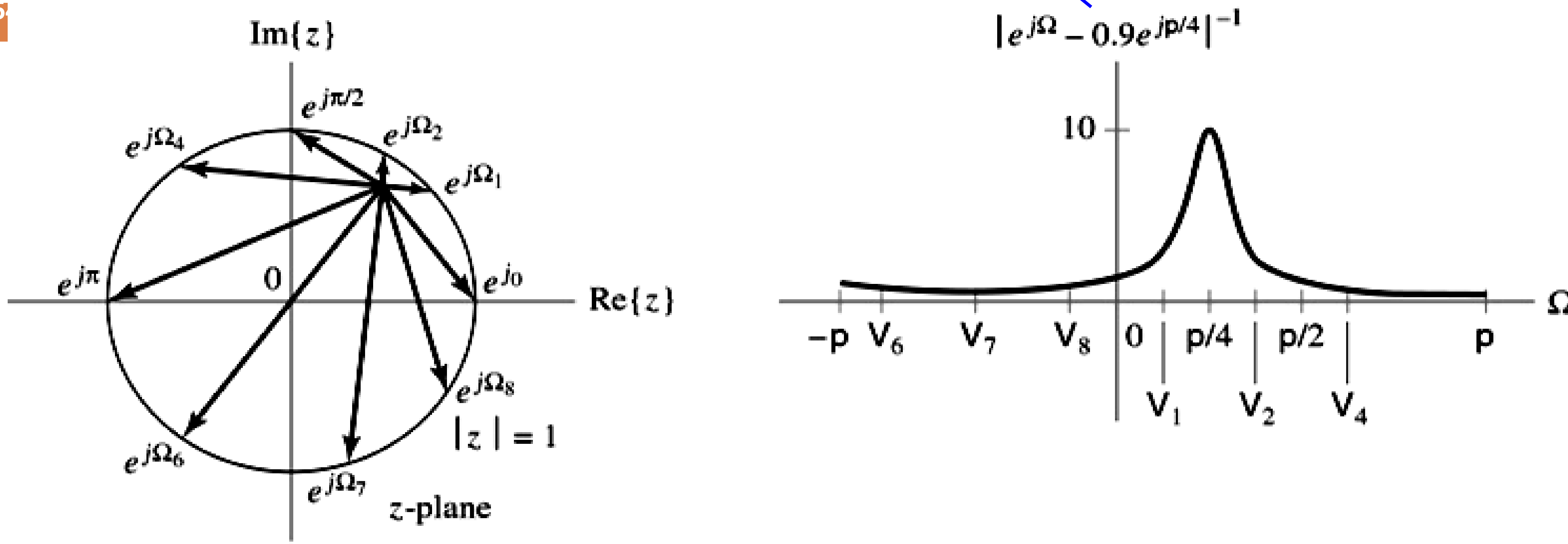
(a)

p ≡ π; v ≡ Ω



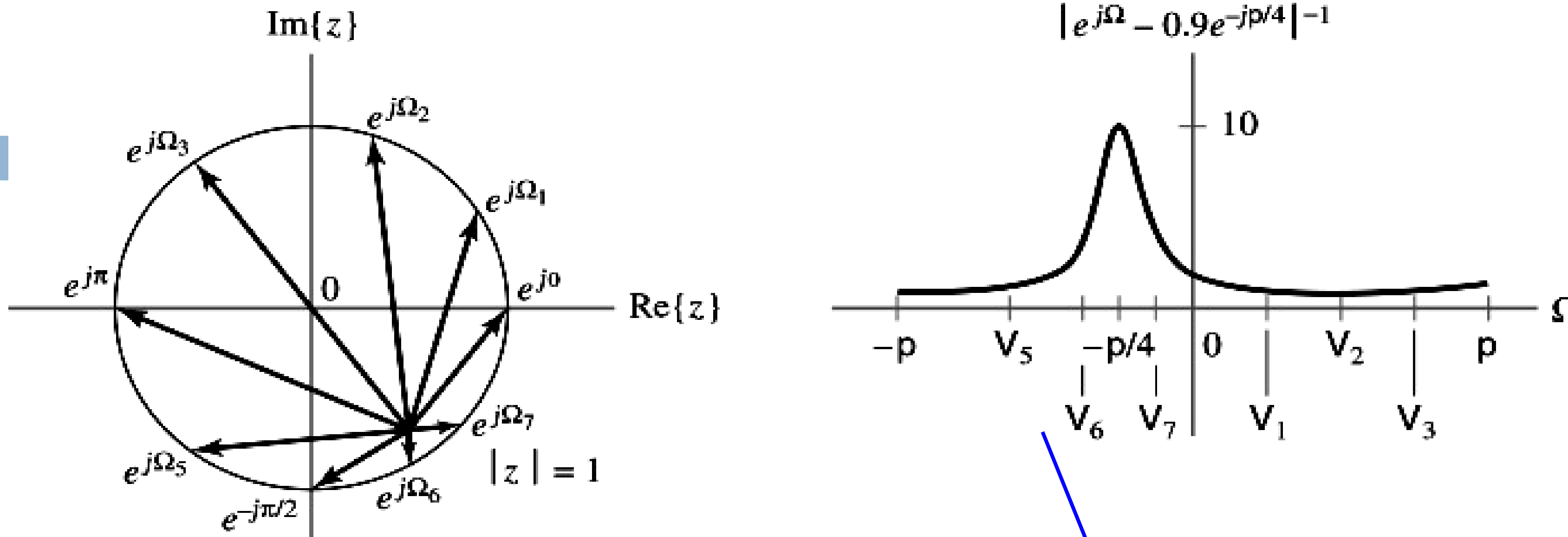
(b)

p ≡ π; v ≡ Ω

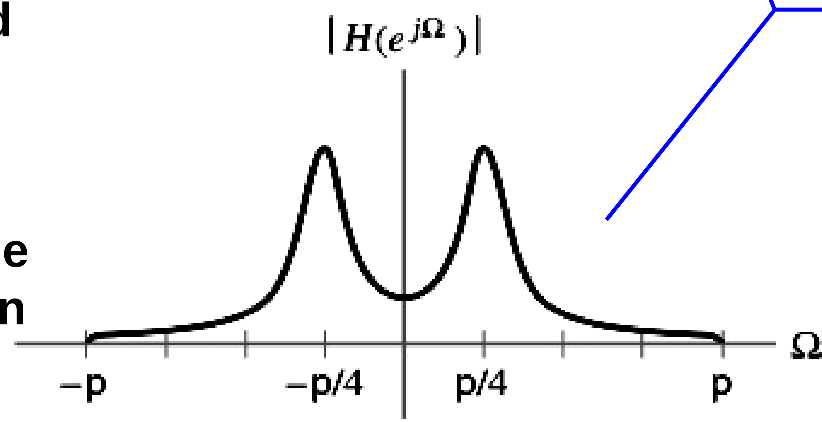


(c)

Figure 7.23a (p. 592)
Solution for Example 7.21.
(c) The component of the magnitude response associated with the pole at $z = e^{j\pi/4}$ is the inverse of the length of a vector from the pole to $e^{j\Omega}$.



(d)



(e)

$p \equiv \pi$; $v \equiv \Omega$

Figure 7.23b (p. 593, continued) (d) The component of the magnitude response associated with the pole at is the inverse of the length of a vector from the pole to $e^{j\Omega}$. (e) The system magnitude response is the product of the response in parts (b)–(d).

Determining the Frequency Response from Poles and Zeros

- **The phase of $|H(e^{j\Omega})|$ may also be evaluated in terms of the phase associated with each pole and zero.**

1. Using Eq.(7.25), we obtain

$$\arg\{H(e^{j\Omega})\} = \arg\{\tilde{b}\} + (N-M)\Omega + \sum_{k=1}^{M-P} \arg\{e^{j\Omega} - c_k\} - \sum_{k=1}^{N-l} \arg\{e^{j\Omega} - d_k\}$$

2. The phase of $H(e^{j\Omega})$ involves the sum of the phase angles due to each zero minus the phase angle due to each pole.

- **Discussion: see pp. 591-594, textbook.**

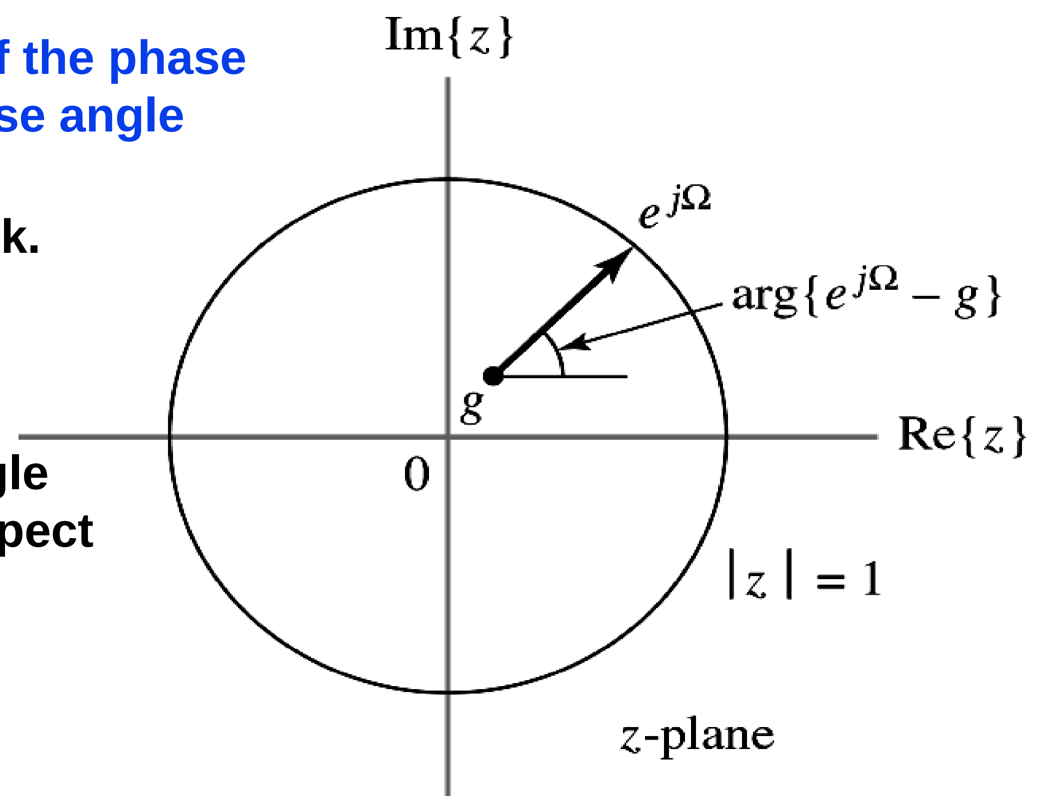
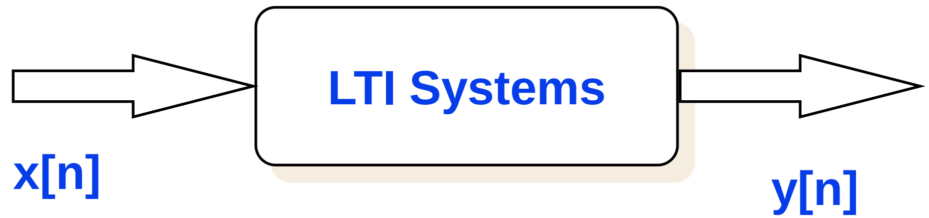


Figure 7.25 (p. 593)
The quantity $\arg\{e^{j\Omega} - g\}$ is the angle of the vector from g to $e^{j\Omega}$ with respect to a horizontal line through g, as shown here.

Remarks

- Four System Descriptions
  - Impulse Response
  - Difference Equations
  - Frequency Response
  - Z-Transform (Transfer Function and System Function)



$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$Y(z) = H(z)X(z)$$

PSPLAB

Remarks

- Introduction
- z Transform
- Unilateral z Transform
- Properties Unilateral z Transform
- Inversion of Unilateral z Transform
- Determining the Frequency Response from Poles and Zeros