Chapter 2 Time-Domain Representations of LTI Systems

- Introduction
- Impulse responses of LTI systems
- Linear constant-coefficients differential or difference equations of LTI systems
- Block diagram representations of LTI systems
- State-variable descriptions for LTI systems
- Summary

Contents

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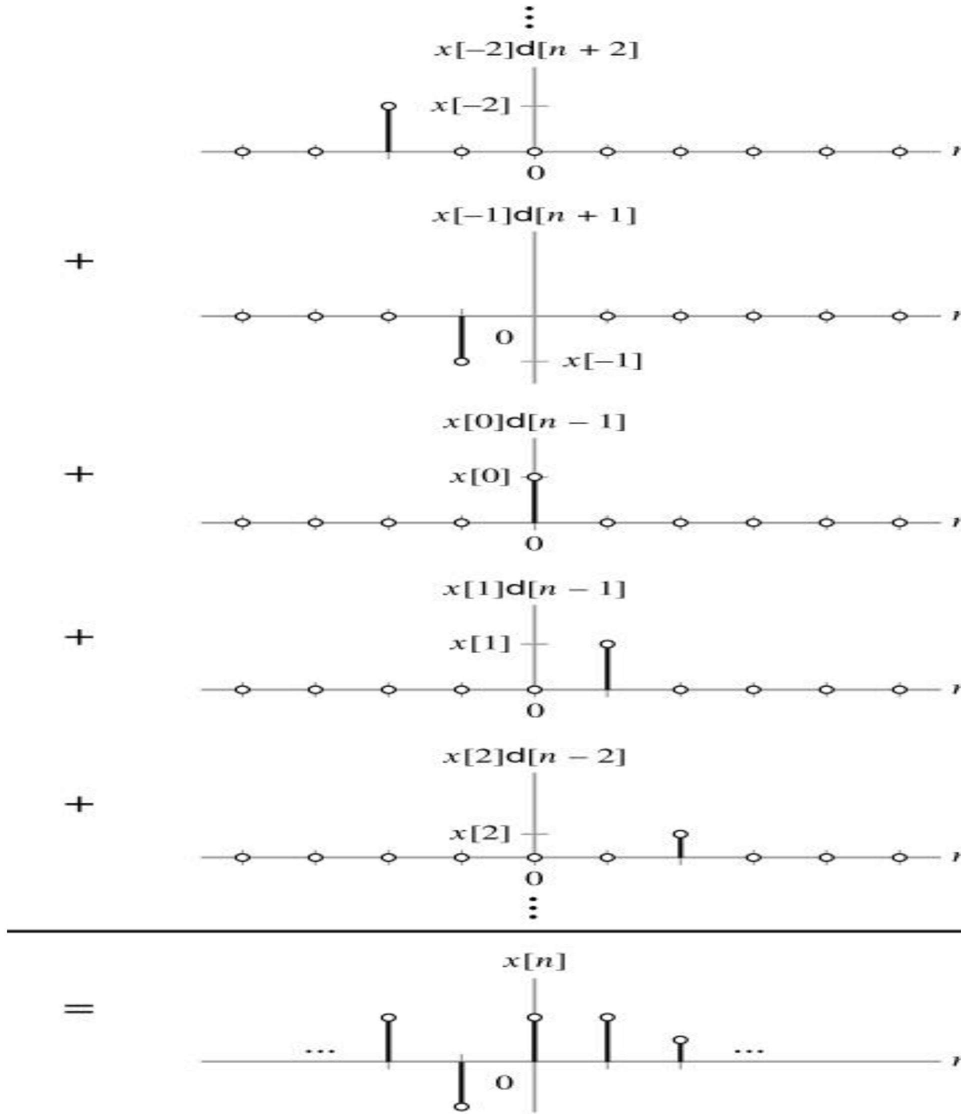
1. Contents (c.1)

- Differential and Difference Equation Representations of LTI Systems
- Solving Differential and Difference Equations
- Characteristics of Systems Described by Differential and Difference Equations
- Block Diagram Representations
- State-Variable Descriptions of LTI Systems.
- Exploring Concepts woth MatLab

2. Convolution Sum

- An arbitrary signal is expressed as a weighted superposition of shifted impulses.

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$



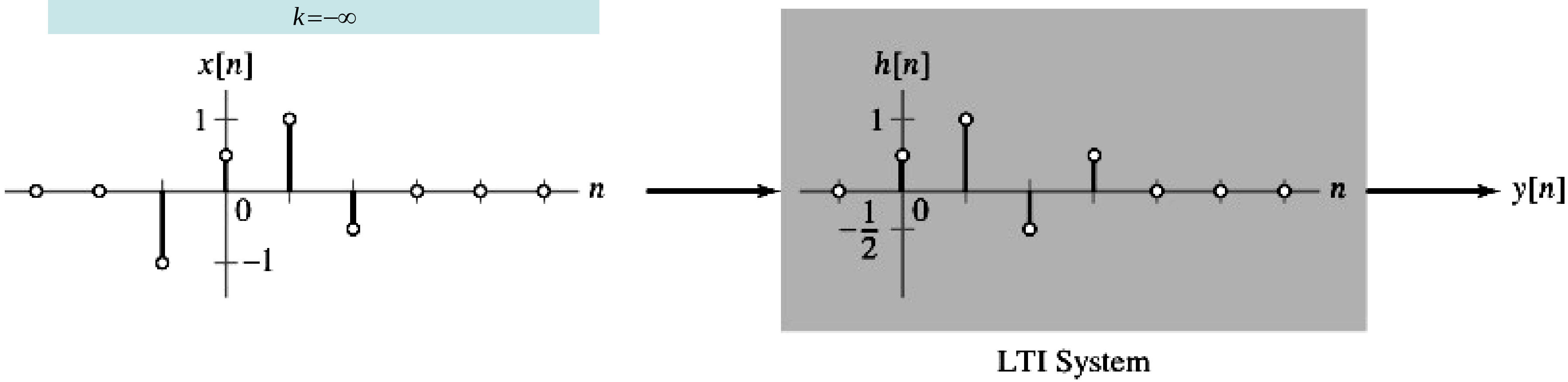
2. Convolution Sum

- Convolution

Impulse Response of the System

$$H[x(n)] = H[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)] = \sum_{k=-\infty}^{\infty} x(k)H[\delta(n-k)] \equiv \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



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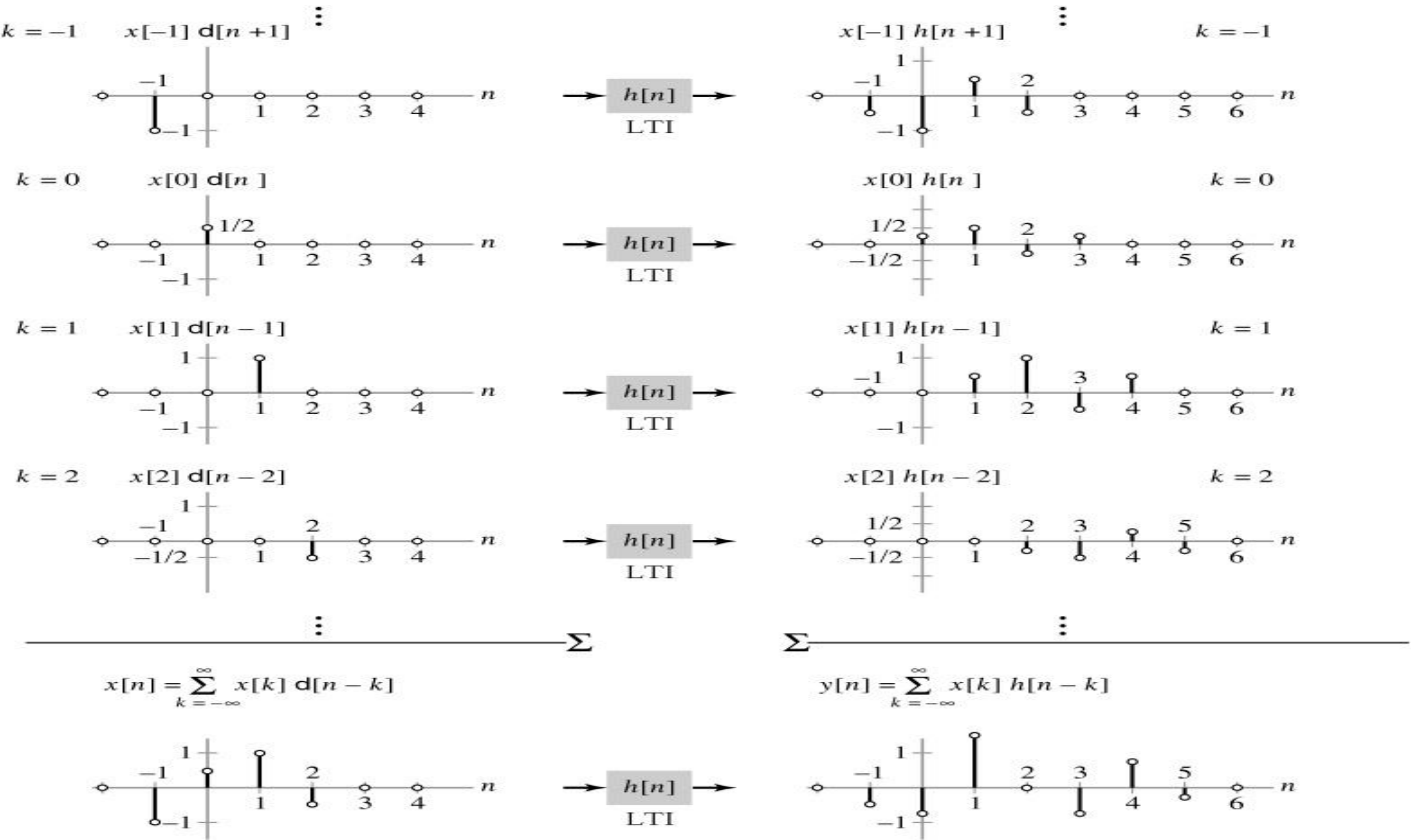
2. Convolution Sum



2. Convolution Sum

$$x[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow y[n]$$

$$x[0]\cdot\delta[n] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_0[n]\cdot x[0]$$

$$x[1]\cdot\delta[n-1] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_1[n]\cdot x[1]$$

$$x[2]\cdot\delta[n-2] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_2[n]\cdot x[2]$$

$$\vdots$$

$$x[k]\cdot\delta[n-k] \longrightarrow \boxed{\text{Linear System}} \longrightarrow h_k[n]\cdot x[k]$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

2. Convolution Sum

Example 2.1 ***Multipath Communication Channel: Direct Evaluation of the Convolution Sum***

Consider the discrete-time LTI system model representing a two-path propagation channel described in Section 1.10. If the strength of the indirect path is a = ½, then

$$y[n] = x[n] + \frac{1}{2}x[n-1]$$

Letting $x[n] = \delta[n]$, we find that the impulse response is

$$h[n] = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} 2, & n = 0 \\ 4, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

2. Convolution Sum

Input = 0 for *n* < 0 and *n* > 0

<Sol.>

1. Input: $x[n] = 2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$

2. Since ***time-shifted impulse input*** $\gamma\delta[n-k]$ ⇒ ***time-shifted impulse response output*** $\gamma h[n-k]$

3. Output:

$$y[n] = 2h[n] + 4h[n-1] - 2h[n-2]$$

$$y[n] = \begin{cases} 0, & n < 0 \\ 2, & n = 0 \\ 5, & n = 1 \\ 0, & n = 2 \\ -1, & n = 3 \\ 0, & n \geq 4 \end{cases}$$

3. Convolution Sum Evaluation Procedure

- Define *intermediate signal*

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

***k* = independent variable**

$$\omega_n[k]=x[k]h[n-k]$$

  - *n* is treated as a constant by writing *n* as a subscript on *w*.
  - $h[n-k]=h[-(k-n)]$ is a reflected (because of $-k$) and time-shifted (by $-n$) version of $h[k]$.
  - Since

$$y[n]=\sum_{k=-\infty}^{\infty}\omega_n[k]$$

  - The time shift *n* determines the time at which we evaluate the output of the system.

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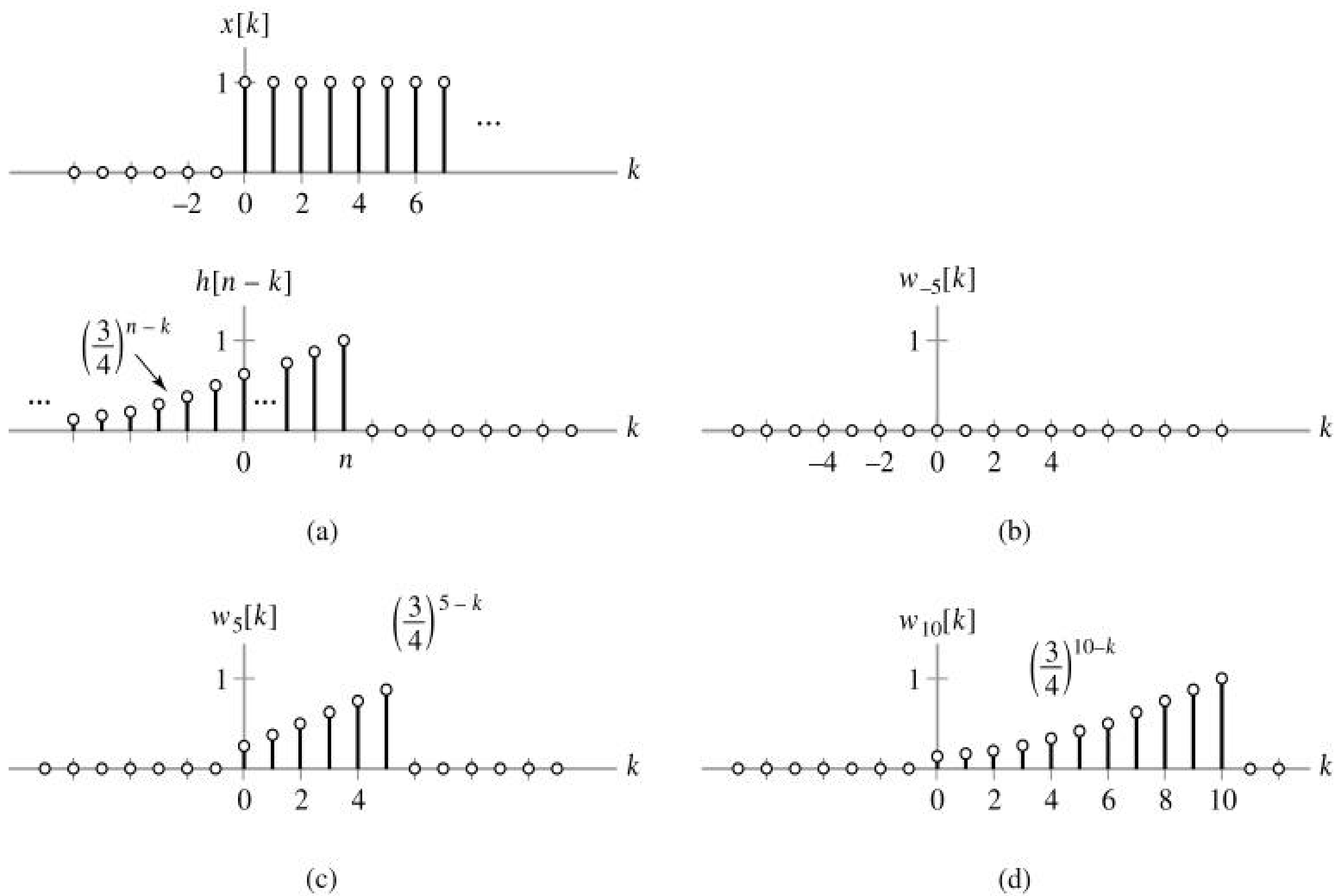
3. Convolution Sum Evaluation Procedure



3. Convolution Sum Evaluation Procedure

- **Procedure (reflect and shift convolution sum evaluation)**

  Step 1: Time-reverse (reflect): *h[k]*   *h[-k]*

  Step 2: Choose an *n value and shift h[.] by n: h[n-k].*

  Step 3: Compute *wn[k] = x[k]h[n-k]*

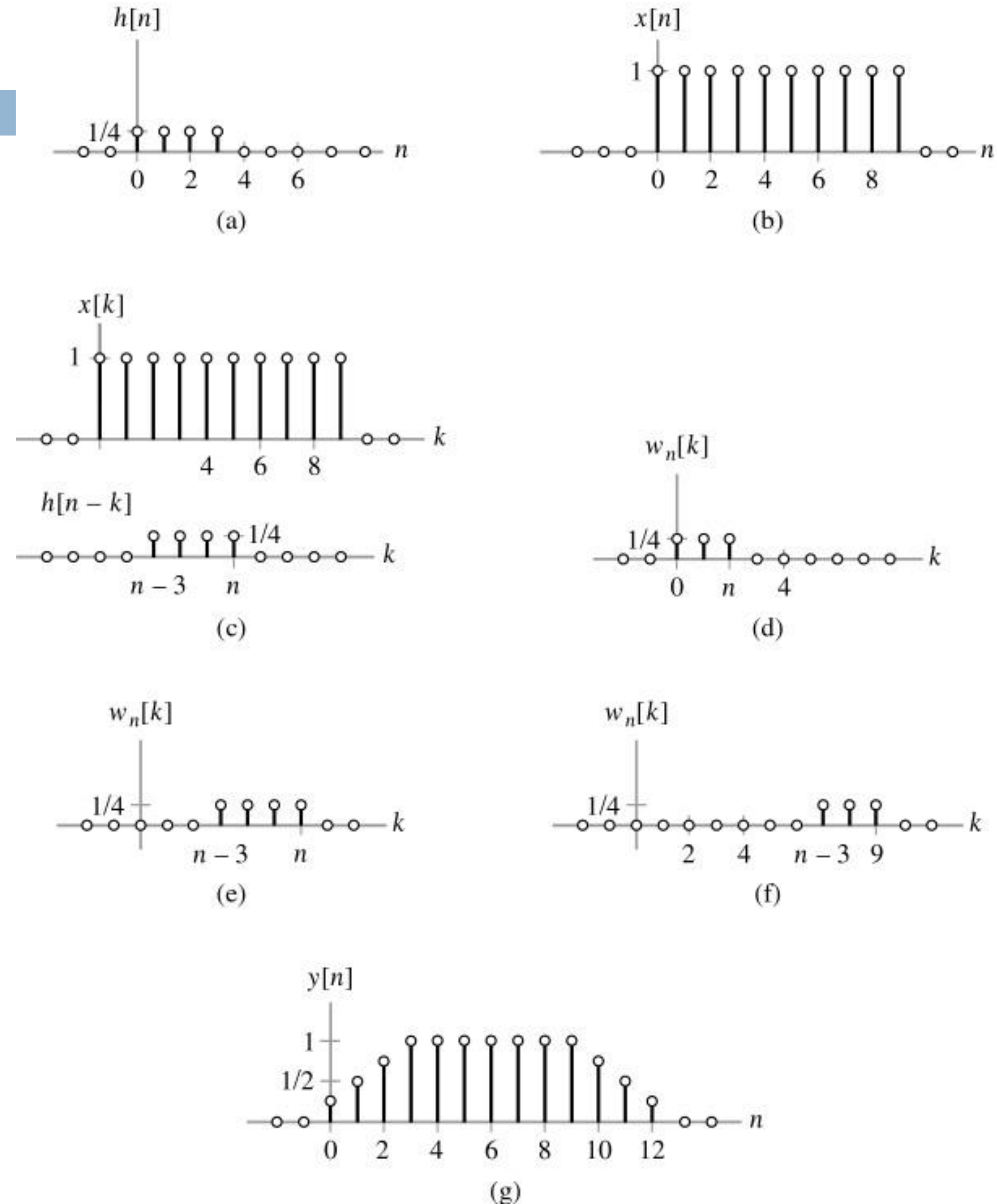
  Step 4: Summation over *k: y[n]=Σk wn[k]*

  Step 5: Choose another *n value, go to Step 2.*

  Step 6: Slide a window of *h[k] over the input signal from left to right.*

3. Convolution Sum Evaluation Procedure

- Check values in y[n]



4. Convolution Integral

- The output of a continuous-time (CT) LTI system may also be determined solely from knowledge of the input and the system's impulse response.
- Signal Integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$

- Linear System

$$y(t) = H\{x(t)\} = H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)H\{\delta(t - \tau)\}d\tau$$

- Shift-Invariance

$$H\{\delta(t - \tau)\} = h(t - \tau)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

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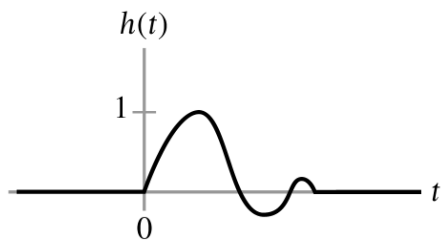
4. Convolution Integral

- Convolution Operator

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau$$



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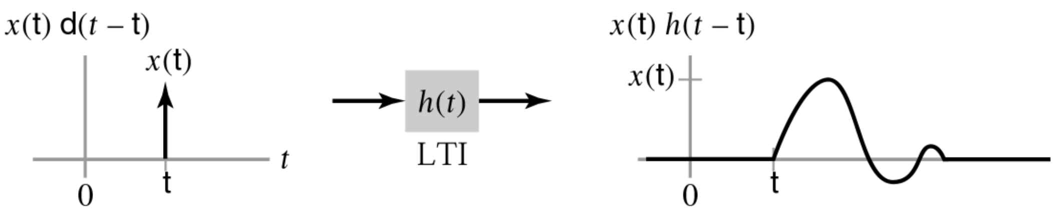

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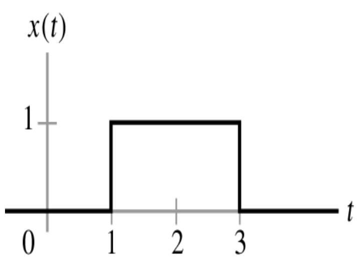
5. Convolution Integral Evaluation Procedure

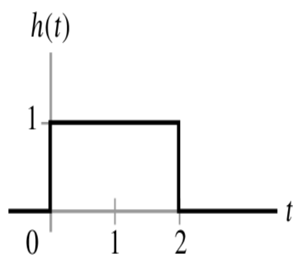
- Define the immediate signal

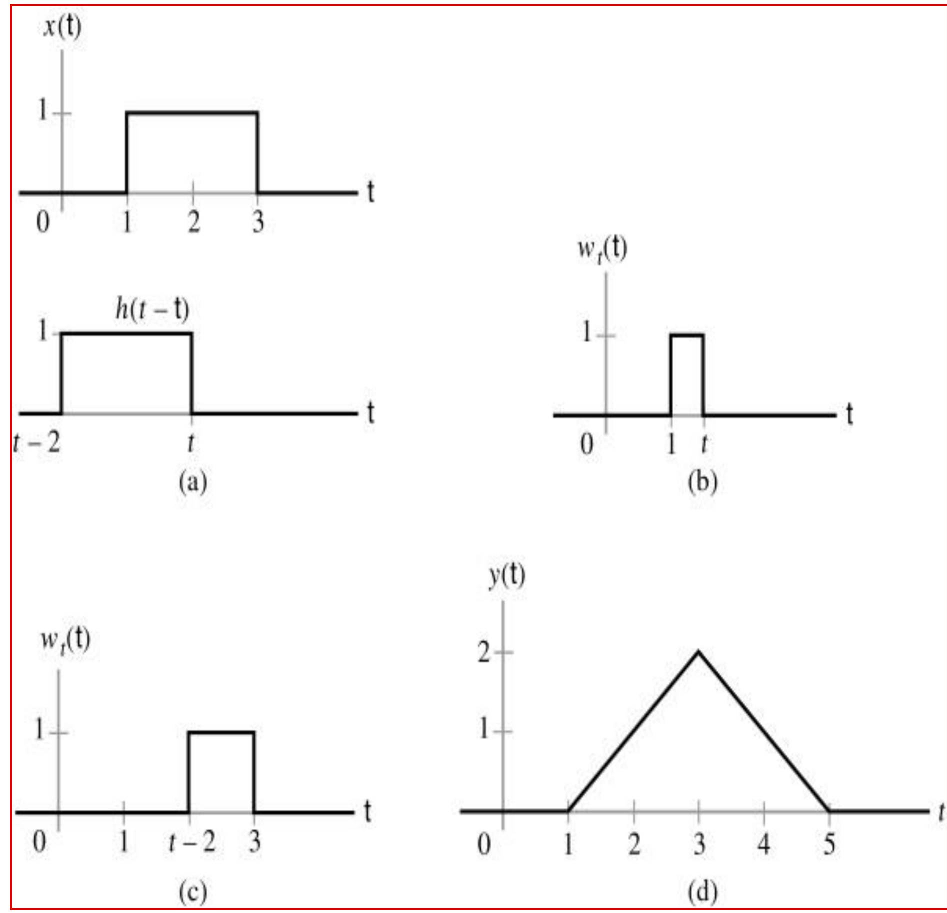
$$w_t(\tau) = x(\tau)h(t-\tau)$$

- Evaluate the output signal at a specific time t

$$y(t) = \int_{-\infty}^{\infty} w_t(\tau)d\tau$$







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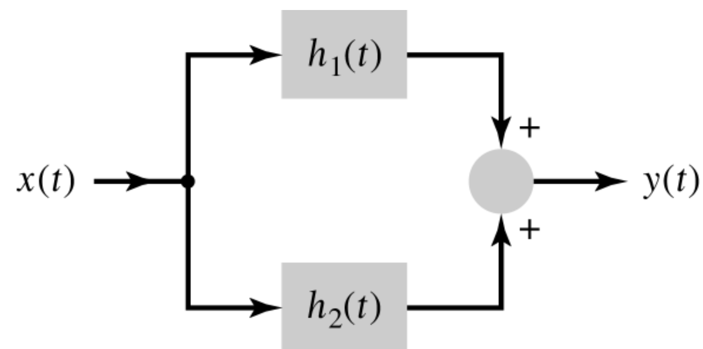
6. Interconnection of LTI Systems

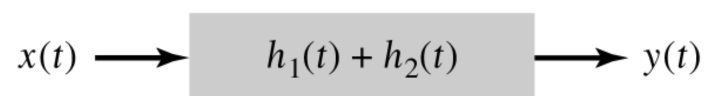
- Relationships between "the impulse response of an interconnection of LTI system" and "the impulse responses of the constituent systems".

$$y(t) = y_1(t) + y_2(t)$$
$$= x(t) * h_1(t) + x(t) * h_2(t)$$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau$$

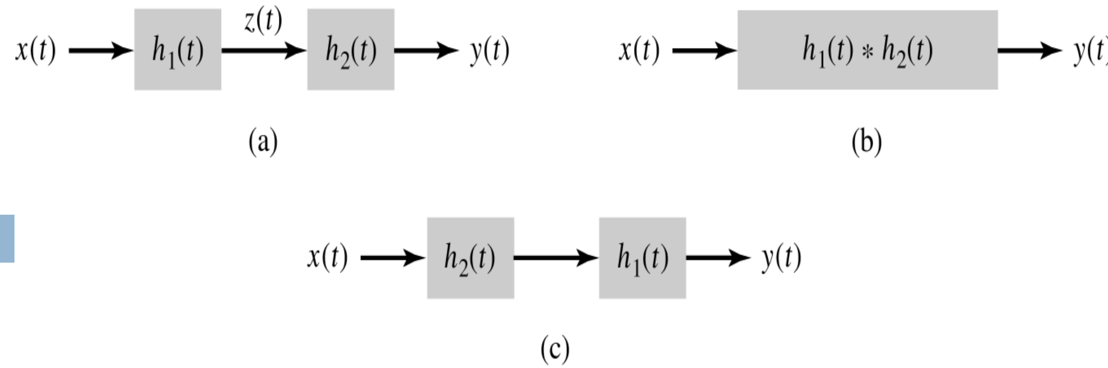
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\{h_1(t-\tau) + h_2(t-\tau)\}d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$



$$\mathrm{x(t) * h_1(t) + x(t) * h_2(t) = x(t) * \{h_1(t) + h_2(t)\}}$$

x[n] * h1[n] + x[n] * h2[n] = x[n] * {h1[n] + h2[n]}

6. Interconnection of LTI Systems



- Cascade
  - $y(t) = z(t) * h2(t) = \{x(t) * h1(t)\} * h2(t)$

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\nu) h_1(\tau - \nu) h_2(t - \tau)\, d\nu d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\nu) \left[\int_{-\infty}^{\infty} h_1(\eta) h_2(t - \nu - \eta)\, d\eta \right] d\nu$$

$$h(t - \nu) = \int_{-\infty}^{\infty} h_1(\eta) h_2(t - \nu - \eta)\, d\eta$$

$$y(t) = \int_{-\infty}^{\infty} x(\nu) h(t - \nu)\, d\nu$$

$$= x(t) * h(t).$$

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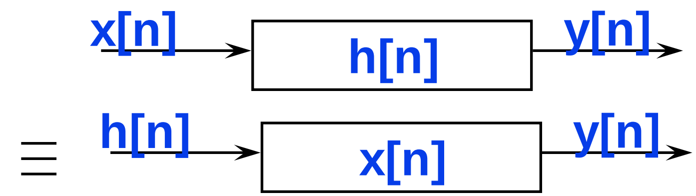
6. Interconnection of LTI Systems

- Communicative

$$y[k] = \sum_{i=-\infty}^{\infty} h[k-i]u[i] \equiv h[k] * u[k]$$

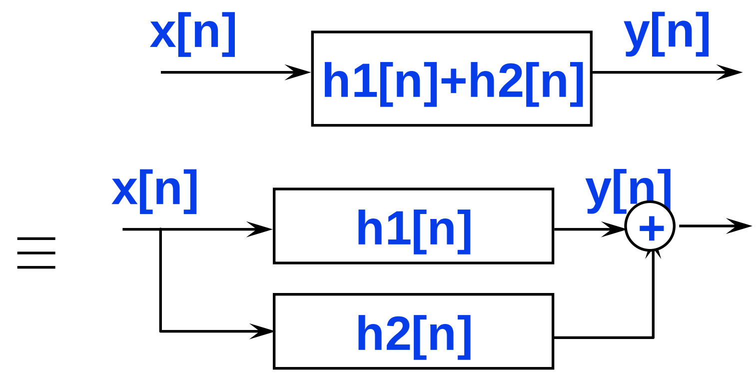
$$= \sum_{i=-\infty}^{\infty} h[i]u[k-i] \equiv u[k] * h[k]$$



- Parallel Sum

$$x[n] * \{h_1[n] + h_2[n]\}$$

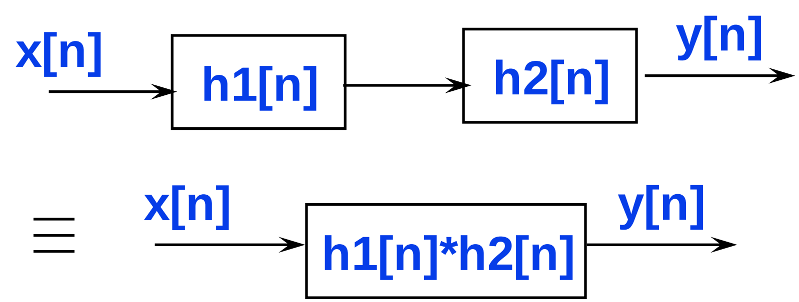
$$= x[n] * h_1[n] + x[n] * h_2[n]$$



- Cascade Form

$$x[n] * \{h_1[n] * h_2[n]\}$$

$$= \{x[n] * h_1[n]\} * h_2[n]$$



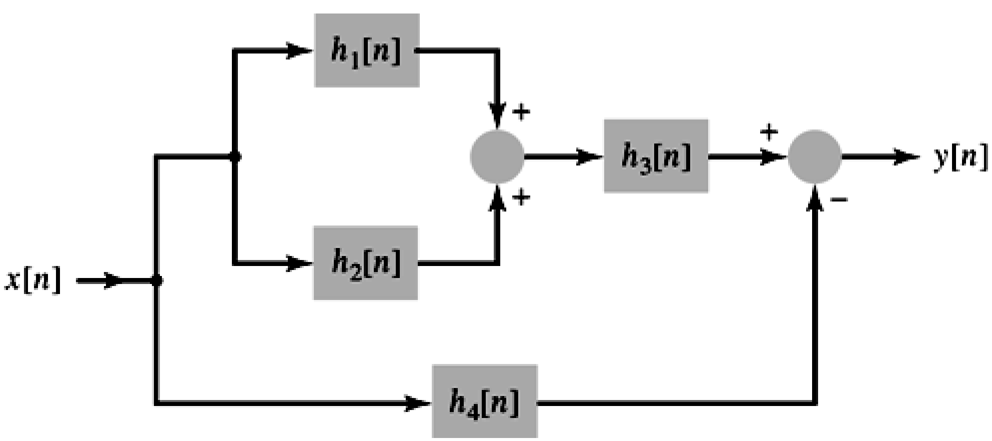
6. Interconnection of LTI Systems

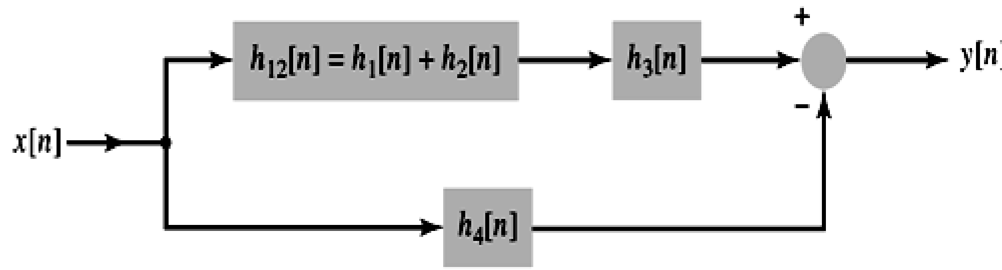
$$x(t)*\{h_1(t)*h_2(t)\}=x(t)*\{h_2(t)*h_1(t)\},$$

$$h_1(t)*h_2(t)=h_2(t)*h_1(t)$$

$$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$$

$$h_1[n]*h_2[n]=h_2[n]*h_1[n]$$





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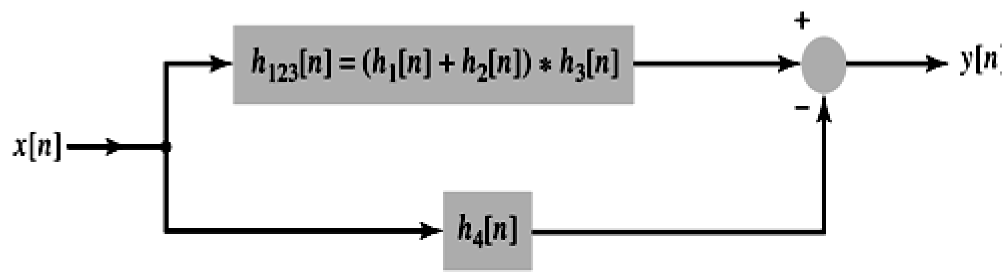

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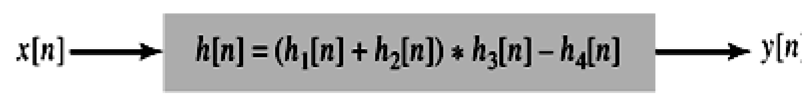

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6. Interconnection of LTI Systems

Table 2.1 Interconnection Properties for LTI Systems

Property	Continuous-time system	Discrete-time system
Distributive	$x(t)*h_1(t)+x(t)*h_2(t)=$ $x(t)*\{h_1(t)+h_2(t)\}$	$x[n]*h_1[n]+x[n]*h_2[n]=$ $x[n]*\{h_1[n]+h_2[n]\}$
Associative	$\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\}$	$\{x[n]*h_1[n]\}*h_2[n]=x[n]*\{h_1[n]*h_2[n]\}$
Commutative	$h_1(t)*h_2(t)=h_2(t)*h_1(t)$	$h_1[n]*h_2[n]=h_2[n]*h_1[n]$

7. Relations between LTI System Properties and the Impulse Response

- Memoryless Systems
  - Discrete-Time Systems

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\boxed{h(\tau) = c\delta(\tau)}$$

  - Continuous-Time Systems

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$\boxed{h[k] = c\delta[k]}$$

7. Relations between LTI System Properties and the Impulse Response

- Causal Systems

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$\boxed{h[k] = 0 \quad \text{for} \quad k < 0}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$

$$\boxed{h(\tau) = 0 \quad \text{for} \quad \tau < 0}$$

7. Relations between LTI System Properties and the Impulse Response

- Stable LTI Systems
  - DT

$$|x[n]| \le M_x \le \infty \Rightarrow \text{Output: } |y[n]| \le M_y \le \infty$$

$$|y[n]| = |h[n] * x[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|$$

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$|a+b| \le |a| + |b|$$

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$|ab| = |a||b|$$

$$|x[n]| \le M_x \le \infty \Rightarrow |y[n]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty.} \Rightarrow \text{BIBO}$$

The impulse response is "absolutely summable" is both sufficient and necessary condition.

7. Relations between LTI System Properties and the Impulse Response

- Stability of LTI Systems (BIBO, Bounded-Input-Bounded Output System)
  - Linear time-invariant systems are stable if and only if the impulse response is absolutely summable, i.e., if

$$\boxed{S=\sum_{k=-\infty}^{\infty}\left|h[k]\right|<\infty}$$

<pf>

Since that

$$\left|y[n]\right|=\left|\sum_{k=-\infty}^{\infty}h[k]x[n-k]\right|$$

$$\leq\sum_{k=-\infty}^{\infty}\left|h[k]\right|\left|x[n-k]\right|$$

- If x[n] is bounded so that $\left|x[n]\right|\leq B_x$
- then $\left|y[n]\right|\leq B_x\sum_{k=-\infty}^{\infty}\left|h[k]\right|$

If S=∞ then the bounded input

$$x[n]=\begin{cases}\dfrac{h^*[-n]}{\left|h[-n]\right|}, & h[n]\neq 0\\ 0, & h[n]=0\end{cases}$$

will generate

$$y[0]=\sum_{k=-\infty}^{\infty}x[-k]h[k]=\sum_{k=-\infty}^{\infty}\frac{\left|h[k]\right|^2}{\left|h[k]\right|}=S$$

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7. Relations between LTI System Properties and the Impulse Response

- Similarly, a continuous-time LTI system is BIBO stable if and only if the impulse response is absolutely integrable

$$\boxed{\int_0^\infty |h(\tau)|\, d\tau < \infty.}$$

Example 2.12 ***Properties of the First-Order Recursive System***

The first-order system is described by the difference equation

$$y[n] = \rho y[n-1] + x[n]$$

and has the impulse response

$$h[n] = \rho^n u[n]$$

Is this system causal, memoryless, and BIBO stable?

<Sol.>

1. The system is causal, since $h[n] = 0$ for $n < 0$.

2. The system is not memoryless, since $h[n] \neq 0$ for $n > 0$.

3. Stability: Checking whether the impulse response is absolutely summable?

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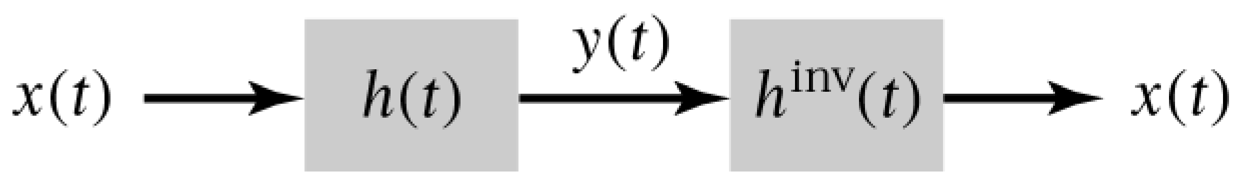
7. Relations between LTI System Properties and the Impulse Response

- **A system is *invertible***
  - If the input to the system can be recovered from the output except for a constant scale factor.
  - The existence of an inverse system that takes the output of the original system as its input and produces the input of the original system.

$$x(t) * (h(t) * h_{inv}(t)) = x(t).$$

$$h(t) * h_{inv}(t) = \delta(t)$$

  - **Similarly,** $h[n] * h_{inv}[n] = \delta[n]$



7. Relations between LTI System Properties and the Impulse Response

Example 2.13 ***Multipath Communication Channels: Compensation by means of an Inverse System***

Consider designing a discrete-time inverse system to eliminate the distortion associated with multipath propagation in a data transmission problem. Assume that a discrete-time model for a two-path communication channel is

$$y[n] = x[n] + ax[n-1].$$

Find a causal inverse system that recovers $x[n]$ from $y[n]$. Check whether this inverse system is stable.

<Sol.>

1. Impulse response:

$$h[n] = \begin{cases} 1, & n = 0 \\ a, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

2. The inverse system $h^{inv}[n]$ must satisfy $h[n] * h^{inv}[n] = \delta[n]$.

$$\Rightarrow \quad h^{inv}[n] + ah^{inv}[n-1] = \delta[n].$$

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7. Relations between LTI System Properties and the Impulse Response

1) For $n < 0$, we must have $h^{inv}[n] = 0$ in order to obtain a causal inverse system

2) For $n = 0$, $\delta[n] = 1$, and eq. (2.32) implies that

$$h^{inv}[n] + ah^{inv}[n-1] = 0,$$

$$h^{inv}[n] = -ah^{inv}[n-1] \qquad (2.33)$$

3. Since $h^{inv}[0] = 1$, Eq. (2.33) implies that $h^{inv}[1] = -a$, $h^{inv}[2] = a^2$, $h^{inv}[3] = -a^3$, and so on.
   The inverse system has the impulse response

$$h^{inv}[n] = (-a)^n u[n]$$

4. To check for stability, we determine whether $h^{inv}[n]$ is absolutely summable, which will be the case if

$$\sum_{k=-\infty}^{\infty} \left|h^{inv}[k]\right| = \sum_{k=-\infty}^{\infty} |a|^k$$ is finite.

♣ For $|a| < 1$, the system is stable.

PSPLAB

7. Relations between LTI System Properties and the Impulse Response

Table 2.2 *Properties of the Impulse Response Representation for LTI Systems*

Property	Continuous-time system	Discrete-time system
Memoryless	$h(t) = c\delta(t)$	$h[n] = c\delta[n]$
Causal	$h(t) = 0 \quad \text{for} \quad t < 0$	$h[n] = 0 \quad \text{for} \quad n < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t) \rvert dt < \infty$	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert < \infty$
Invertibility	$h(t) * h^{inv} = \delta(t)$	$h[n] * h^{inv}[n] = \delta[n]$

8. Step Response

- Step response is the output due to a unit step input signal
  - *Step input signals are often used to characterize the response of an LTI system to sudden changes in the input*

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k].$$

  - Since $u[n-k] = 0$ for $k > n$ and $u[n-k] = 1$ for $k \leq n$, we have

$$\boxed{s[n] = \sum_{k=-\infty}^{n} h[k].}$$

$$h[n] = s[n] - s[n-1]$$

  - *Similarly for CT system*

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$h(t) = \frac{d}{dt}s(t)$$

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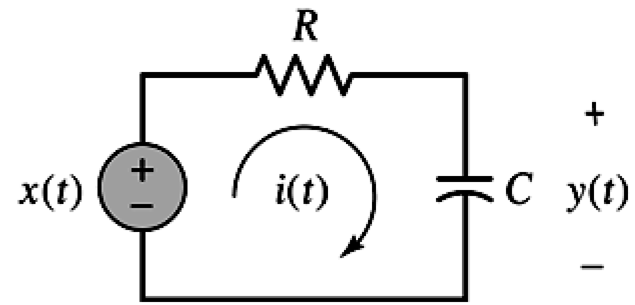
8. Step Response

Example 2.14 ***RC*** **Circuit: Step Response**

The impulse response of the *RC* circuit depicted in Fig. 2.12 is

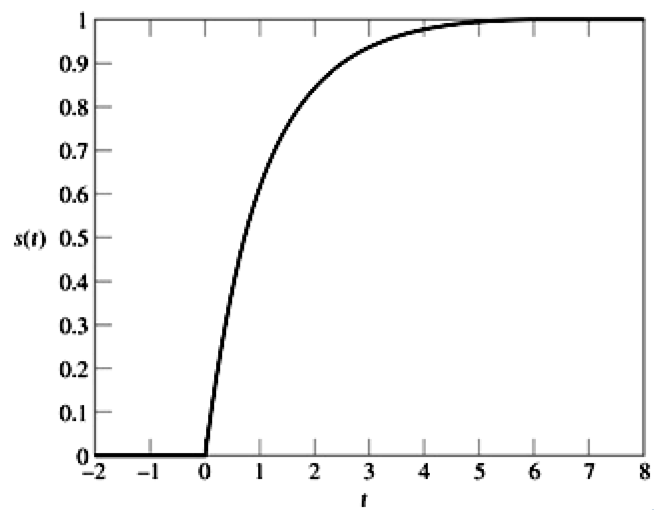
$$h(t)=\frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$



1. Step respose:

$$s(t)=\int_{-\infty}^{t}\frac{1}{RC}e^{-\frac{\tau}{RC}}u(\tau)d\tau.$$

$$s(t)=\begin{cases}0, & t<0\\ \dfrac{1}{RC}\displaystyle\int_{-\infty}^{t}e^{-\frac{\tau}{RC}}u(\tau)d\tau & t\geq 0\end{cases}$$



$$s(t)=\begin{cases}0, & t<0\\ \dfrac{1}{RC}\displaystyle\int_{0}^{t}e^{-\frac{\tau}{RC}}d\tau & t\geq 0\end{cases}$$

$$=\begin{cases}0, & t<0\\ 1-e^{-\frac{t}{RC}}, & t\geq 0\end{cases}$$

9. Differential and Difference Equation Representations of LTI Systems

- **Linear constant-coefficient difference and differential equations provide another representation for the input-output characteristics of LTI systems.**
- ***CT: Constant coefficient* differential equation**

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

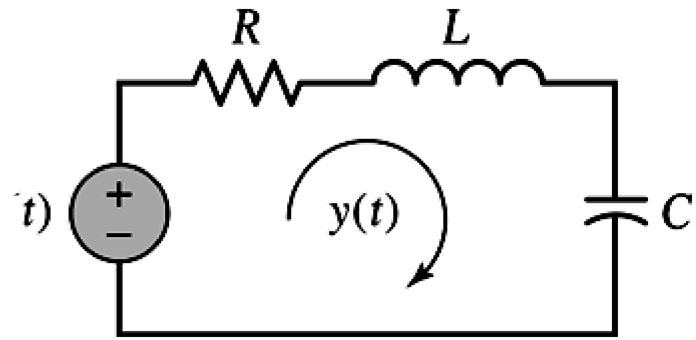
- ***DT: Constant coefficient* difference equation**

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The *order of the differential or difference equation is* (N,M), representing the number of energy storage devices in the system. Often, N>= M, and the order is described using only N.

9. Differential and Difference Equation Representations of LTI Systems

- Example
  - Summing the voltage loop



$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$

  - Differentiating both sides

$$\frac{1}{C}y(t) + R\frac{d}{dt}y(t) + L\frac{d^2}{dt^2}y(t) = \frac{d}{dt}x(t)$$

- The order is N = 2 and the circuit contains two energy storage devices: a capacitor and an inductor.

9. Differential and Difference Equation Representations of LTI Systems

- Example of a Difference Equation

$$y[n]+y[n-1]+\frac{1}{4}y[n-2]=x[n]+2x[n-1]$$

  - Rearrange the equation

$$y[n]=x[n]+2x[n-1]-y[n-1]-\frac{1}{4}y[n-2]$$

  - Starting from $n=0$, *compute the current output from the input and* the past outputs

$$y[0]=x[0]+2x[-1]-y[-1]-\frac{1}{4}y[-2],$$
$$y[1]=x[1]+2x[0]-y[0]-\frac{1}{4}y[-1],$$
$$y[2]=x[2]+2x[1]-y[1]-\frac{1}{4}y[0],$$
$$y[3]=x[3]+2x[2]-y[2]-\frac{1}{4}y[1],$$
$$\vdots$$

- **We must know *two most recent past values of the output, namely,* $y[-1]$ and $y[-2]$. *These values are called initial conditions.***

9. Differential and Difference Equation Representations of LTI Systems

- **Initial Conditions**
  - Summarize all the information about the system's past needed to determine future outputs.
    - In general, the number of initial conditions required to determine the output is equal to the maximum memory of the system.
  - DT: Nth-order difference eqn. N values
    - y[-N], y[-N+1], …, y[-1].
  - CT: Nth-order differential eqn. first N derivatives of the output; th

$$y(t)\Big|_{t=0-},\quad \frac{d}{dt}y(t)\Big|_{t=0-},\quad \frac{d^2}{dt^2}y(t)\Big|_{t=0-},\quad \ldots,\quad \frac{d^{N-1}}{dt^{N-1}}y(t)\Big|_{t=0-}$$

  - Note: The textbook says the first N derivatives, which include $y(t)|_{t=0-}$, the 0th-order derivative.

10. Solving Differential and Difference Equations

- **Given *x(t)* (*input*), *find y (t)* (*output*)**
  - ***y* = $y^{(h)}$ + $y^{(p)}$ = *homogeneous solution* + *particular* solution**
- Homogeneous solution for Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y^{(h)}(t) = 0$$

  - The homogeneous solution is the solution the form

$$y^{(h)}(t) = \sum_{i=0}^{N} c_i e^{r_i t}$$

  - c_i *are to be decided in the complete solution and* r_i *are the N* roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

PSPLAB

10. Solving Differential and Difference Equations

- Homogeneous solution for Difference Equation

$$\sum_{k=0}^{N} a_k y^{(h)}[n-k] = 0$$

  - The homogeneous solution is the solution the form

$$y^{(h)}[n] = \sum_{i=1}^{N} c_i r_i^n$$

  - c_i *are to be decided in the complete solution and* r_i *are the N* roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^{N-k} = 0$$

10. Solving Differential and Difference Equations

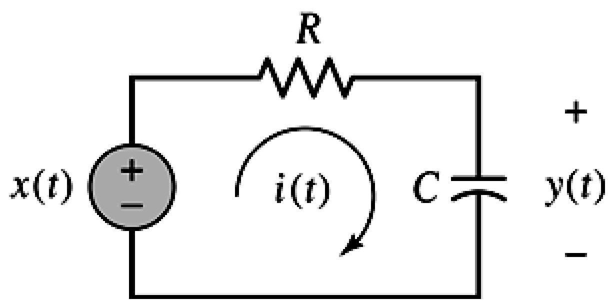
- If a root r_j is repeated p times in characteristic eqs., the corresponding solutions are

  Continuous-time case: $e^{r_j t},\ \ te^{r_j t},\ \ \dots,\ \ t^{p-1}e^{r_j t}$

  Discrete-time case: $r_j^n,\ \ nr_j^n,\ \ \dots,\ \ n^{p-1}r_j^n$

- **Example**

$$y(t)+RC\frac{d}{dt}y(t)=x(t)$$



1. Homogeneous Eq.: $y(t)+RC\frac{d}{dt}y(t)=0$

2. Homo. Sol.: $y^{(h)}(t)=c_1e^{r_1t}\quad \mathrm{V}$

3. Characteristic eq.: $1+RCr_1=0$ ⇒ $r_1 = -1/RC$

4. Homogeneous solution:

$$y^{(h)}(t)=c_1e^{\frac{-t}{RC}}\quad \mathrm{V}$$

10. Solving Differential and Difference Equations

- Example

$$y[n]-\rho y[n-1]=x[n]$$

1. Homogeneous Eq.: $y[n]-\rho y[n-1]=0$

2. Homo. Sol.: $y^{(h)}[n]=c_1 r_1^n$

3. Characteristic eq.: $r_1-\rho=0$

4. Homogeneous solution: $y^{(h)}[n]=c_1\rho^n$

10. Solving Differential and Difference Equations

- **Particular Solution**
  - The particular solution $y^{(p)}$ represents **any solution** of the differential or difference eqn. for the given input.
  - $y^{(p)}$ is not unique. A particular solution is usually obtained by **assuming an output of the same general form** as the input yet is independent of all terms in the homogenous solution.
- Example $y[n]-\rho y[n-1]=x[n]$

if the input is $x[n] = (1/2)^n$.

1. Particular solution form: $y^{(p)}[n]=c_p\left(\frac{1}{2}\right)^n$

2. Substituting $y^{(p)}[n]$ and $x[n]$ into the given difference

$$c_p\left(\tfrac{1}{2}\right)^n-\rho c_p\left(\tfrac{1}{2}\right)^{n-1}=\left(\tfrac{1}{2}\right)^n \qquad c_p(1-2\rho)=1 \qquad y^{(p)}[n]=\frac{1}{1-2\rho}\left(\frac{1}{2}\right)^n$$

10. Solving Differential and Difference Equations

- Hypothesis for the Particular Solution

Table 2.3 ***Form of Particular Solutions Corresponding to Commonly Used Inputs***

Continuous Time		Discrete Time	
Input	Particular Solution	Input	Particular Solution
1	c	1	c
t	$c_1 t + c_2$	n	$c_1 n + c_2$
e^{-at}	ce^{-at}	α^n	$c\alpha^n$
$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

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10. Solving Differential and Difference Equations

- Procedure

Procedure 2.3:* *Solving a Differential or Difference equation

1. Find the form of the homogeneous solution $y^{(h)}$ from the roots of the characteristic equation.
2. Find a particular solution $y^{(p)}$ by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution.
3. Determine the coefficients in the homogeneous solution so that the complete solution $y = y^{(h)} + y^{(p)}$ satisfies the initial conditions.

10. Solving Differential and Difference Equations

- Example

First-Order Recursive System* (Continued)*: Complete Solution

Find the complete solution for the first-order recursive system described by the difference equation $y[n]-\frac{1}{4}y[n-1]=x[n]$

if the input is $x[n]=(1/2)^n u[n]$ and the initial condition is $y[-1]=8$.

<Sol.> **1. Homogeneous sol.:**

$$y^{(h)}[n]=c_1\left(\tfrac{1}{4}\right)^n$$

2. Particular solution:

$$y^{(p)}[n]=2\left(\frac{1}{2}\right)^n$$

3. Complete solution:

$$y[n]=2(\frac{1}{2})^n+c_1(\frac{1}{4})^n$$

4. Coefficient c_1 determined by I.C.:

$$y[0]=x[0]+1/4\,y[-1] \Longrightarrow y[0]=x[0]+(1/4)\times 8=3$$

We substitute $y[0]=3$

$$3=2\left(\frac{1}{2}\right)^0+c_1\left(\frac{1}{4}\right)^0 \Longrightarrow c_1=1$$

5. Final solution:

$$y[n]=2\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n \quad \text{for } n\ge 0$$

10. Solving Differential and Difference Equations

1. Homogeneous sol.: $y^{(h)}(t) = c_1 e^{r_1 t} \quad \text{V}$

2. Particular solution:

$$y^{(p)}(t) = \frac{1}{1+(RC)^2}\cos(t) + \frac{RC}{1+(RC)^2}\sin(t) \quad \text{V}$$

$\omega_0 = 1$

3. Complete solution:

$R = 1\ \Omega$, $C = 1$ F

$$y(t) = ce^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{V}$$

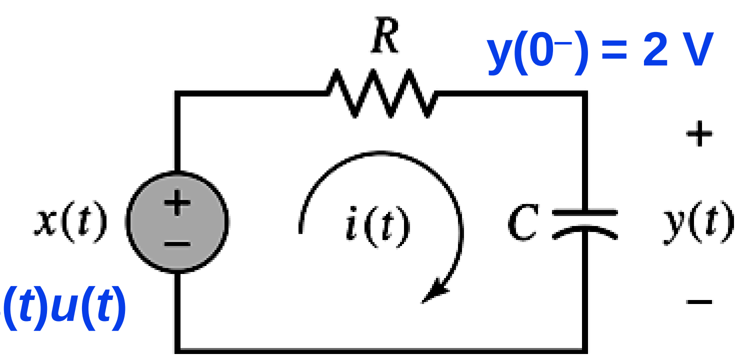
4. Coefficient c_1 determined by I.C.: $y(0^-) = y(0^+)$

$$2 = ce^{-0^+} + \frac{1}{2}\cos 0^+ + \frac{1}{2}\sin 0^+ = c + \frac{1}{2}$$

$c = 3/2$

5. Final solution:

$$y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{V}$$



$y(0^-) = 2$ V

$x(t) = \cos(t)u(t)$

$R = 1\ \Omega$ and $C = 1$ F

11. Characteristics of Systems Described by Differential and Difference Equations

- **Natural Response**
  - The system output for ***zero input***. It is produced by the stored energy or memory of the past (non-zero initial conditions).
  - Homogeneous solution by choosing ***the coefficients ci*** *so that the initial conditions are* satisfied. It does not involve the particular solution.
- The natural response is determined without **translating initial conditions forward in time.**

11. Characteristics of Systems Described by Differential and Difference Equations

- **Forced response: the system output due to the input signal assuming zero initial conditions.**
  - It has the same form as the complete sol.
  - A system with zero initial conditions is said to be "at rest". The at-rest (zero state) initial conditions must be translated forward before solving for the undetermined coefficients.
    - *DT: $y[-N] = y[-N+1] = \ldots = y[-1] = 0$ $\;y[0], y[1], \ldots, y[N-1]$*
    - *CT: Initial conditions at $t = 0-$ $\;t = 0+$*
- **We shall only solve the differential eqns. of which initial conditions at $t = 0+$ *are equal to the* zero initial conditions at $t = 0-$.**

11. Characteristics of Systems Described by Differential and Difference Equations

- Find the natural response of the this system, assuming that y(0) = 2 V, R = 1 Ω and C = 1 F.

$$y(t) + RC\frac{d}{dt}y(t) = x(t)$$

1. Homogeneous sol.: $y^{(h)}(t) = c_1 e^{-t}$ V

2. I.C.: y(0) = 2 V

$\Rightarrow$ $y^{(n)}(0) = 2$ V $\Rightarrow$ $c_1 = 2$

3. Natural Response:

$$y^{(n)}(t) = 2e^{-t} \quad \text{V}$$

11. Characteristics of Systems Described by Differential and Difference Equations

- **Impulse response**

1. We do not know the form of particular sol. for impulse input (why?).
   - **In general, we can find the *step response assuming that the* system is at rest. Then, the impulse response is obtained by taking differentiation (CT) or difference (DT) on the step response.**
     - Step response is the output due to a unit step input signal
2. Impulse response is obtained under the assumption that the systems are initially at rest or the input is known for all time.

11. Characteristics of Systems Described by Differential and Difference Equations

- **Linearity**
  - The *forced response of an LTI system described by a* differential or difference eqn. is linear with respect to theinput (zero I.C.).
  - The *natural response of an LTI system described by a* differential or difference eqn. is linear with respect to the initial conditions (zero input).

11. Characteristics of Systems Described by Differential and Difference Equations

- **Time invariance**
  - The *forced response of an LTI system described by a* differential or difference eqn. is time-invariant.
    - **In general, the output (complete sol.) of an LTI system described by a differential or difference eqn. is *not timeinvariant*** because the initial conditions do not shift in time.
- **Causality**
  - The *forced response (zero I.C.) is causal*

11. Characteristics of Systems Described by Differential and Difference Equations

- **Stability**
  - The natural response (zero input) must be bounded for any set of initial conditions. Hence, each term in the natural response must be bounded.
  - ***DT: is bounded for all i. (When , the natural response*** does not decay, and the system is on the verge of instability.)
    - *A DT LTI system is stable iff all roots have magnitude less than unity.*
  - ***CT: is bounded for all i. (When , the natural response*** does not decay, and the system is on the verge of instability.)
    - *A CT LTI system is stable iff the real parts of all roots are negative.*

11. Characteristics of Systems Described by Differential and Difference Equations

- **Response time (to an input)**
  - The time it takes an LTI system to respond a (input) transient.
  - When the *natural response decays to zero, the system* behavior is governed only by the particular solution, **which is the same as the input. Thus, the response time depends on the roots of characteristic eqn. (It must be a stable system.)**
- ***DT: slowest decay term*** **➔** ***the largest magnitude of*** the characteristic roots
- ***CT: slowest decay term*** **➔** ***the smallest (magnitude)*** negative real-part of the characteristic roots

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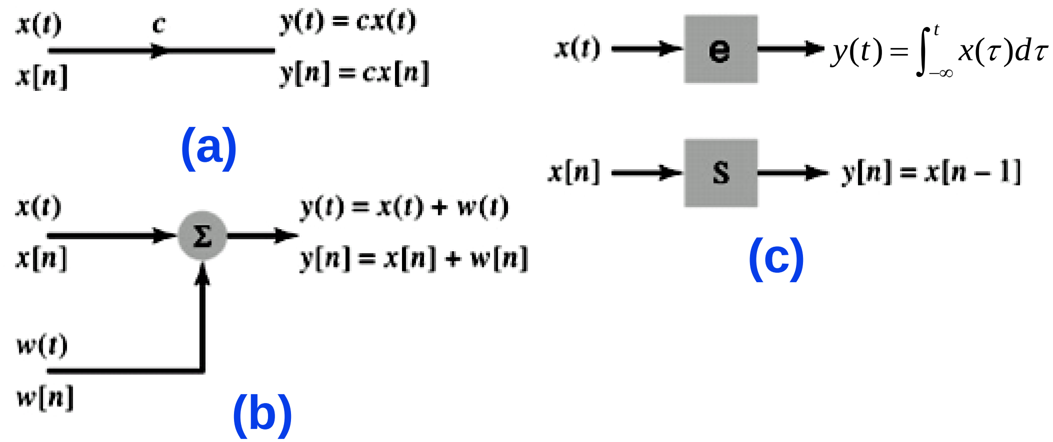
12. Block Diagram Representations

- **Block Diagram Representation**
  - A *block diagram is an interconnection of elementary operations that act on* the input signal. It describes the system's *internal computations* or operations are ordered.
  - (More detailed representation than impulse response or diff. eqns.)
  - The same system may have *different block diagram representations. (Not unique!)*
  - Elementary operations:
    - Scalar multiplication
    - Addition
    - *CT: Integration*
    - *DT: time shift*



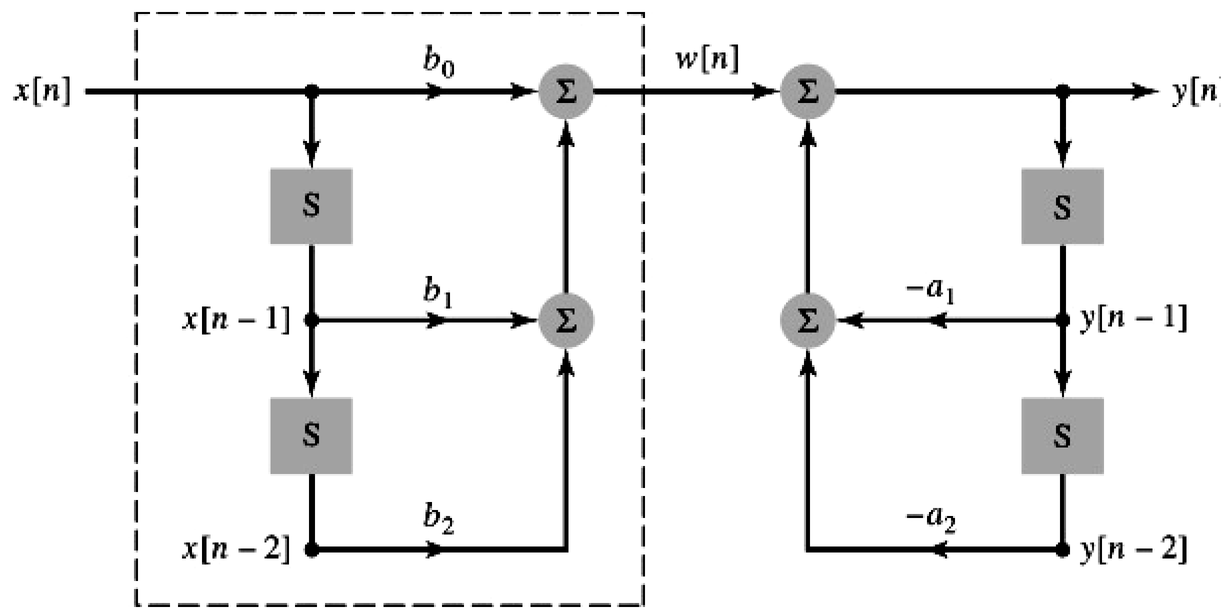
12. Block Diagram Representations

- Direct Form I

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$w[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$y[n] = w[n] - a_1 y[n-1] - a_2 y[n-2]$$



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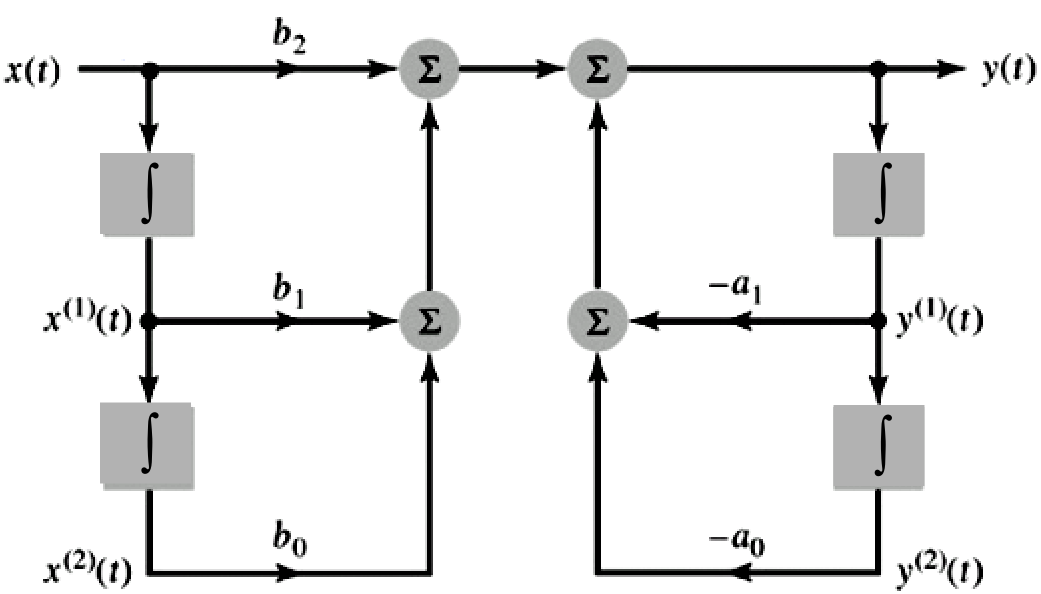
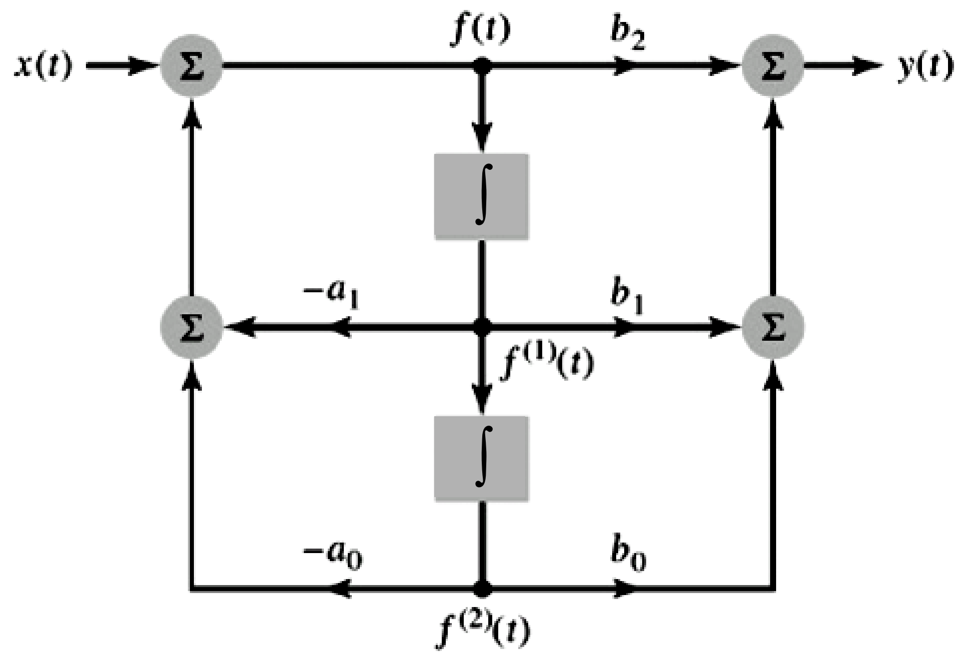
12. Block Diagram Representations

- Direct Form II
  - Interchange their order without changing the input-output behavior of the cascade. We then can merge the two sets of shifts into one.





13. State-Variable Descriptions of LTI Systems.

Matrix form of output equation:

$$y[n] = [b_1 - a_1 \quad b_2 - a_2]\begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + [1]x[n]$$

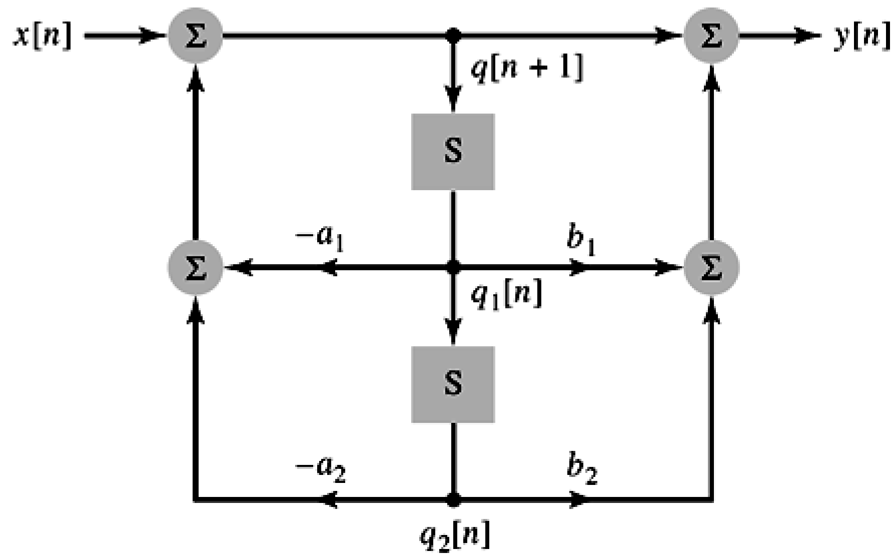
Define state vector as the column vector

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

So

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n]$$



$$\mathbf{A} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{c} = [b_1 - a_1 \quad b_2 - a_2] \quad D = 1$$

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13. State-Variable Descriptions of LTI Systems

- **State-variables are not unique.**
- **Different state-variable descriptions may be obtained by transforming the state variables.**
  - The new state-variables are a weighted sum of the original ones.
  - This changes the form of **A,b,c, and *D, but does not*** change the I/O characteristics of the system.

13. State-Variable Descriptions of LTI Systems

The original state-variable description

1. State equation:

$$q_1[n+1] = \alpha q_1[n] + \delta_1 x[n]$$

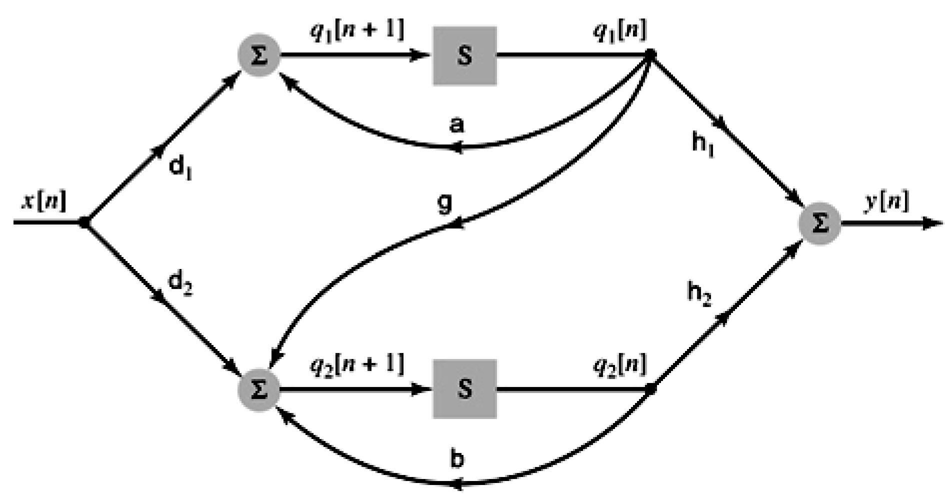
$$q_2[n+1] = \gamma q_1[n] + \beta q_2[n] + \delta_2 x[n]$$

2. Output equation:

$$y[n] = \eta_1 q_1[n] + \eta_2 q_2[n]$$

3. Define state vector as

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$



In standard form of dynamic equation:

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n]$$

$$y[n] = \mathbf{c}\mathbf{q}[n] + Dx[n]$$

$$\mathbf{A} = \begin{bmatrix} \alpha & 0 \\ \gamma & \beta \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

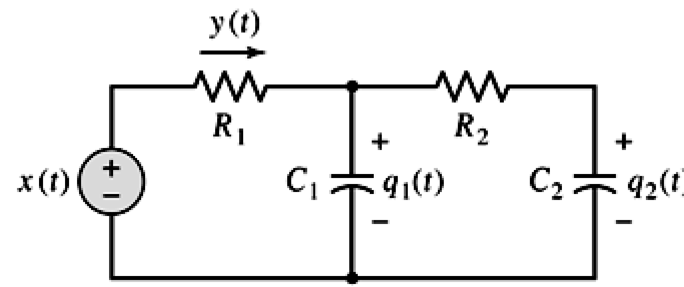
$$\mathbf{c} = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \qquad D = [2]$$

13. State-Variable Descriptions of LTI Systems

- Continuous-Time

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t)$$

$$y(t) = \mathbf{c}\mathbf{q}(t) + Dx(t)$$



1. State variables: The voltage across each capacitor.

2. KVL Eq. for the loop involving *x*(*t*), *R*$_1$, and *C*$_1$:

$$x(t) = y(t)R_1 + q_1(t)$$

$$\Rightarrow \quad y(t) = -\frac{1}{R_1}q_1(t) + \frac{1}{R_1}x(t)$$

Output equation

3. KVL Eq. for the loop involving *C*$_1$, *R*$_2$, and *C*$_2$:

$$q_1(t) = R_2 i_2(t) + q_2(t)$$

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13. State-Variable Descriptions of LTI Systems

4. The current $i_2(t)$ through R_2:

$$i_2(t) = \frac{1}{R_2} q_1(t) - \frac{1}{R_2} q_2(t)$$

eliminate $i_2(t)$

5. KCL Eq. between R_1 and R_2:

$$i_2(t) = C_2 \frac{d}{dt} q_2(t)$$

$$\frac{d}{dt} q_2(t) = \frac{1}{C_2 R_2} q_1(t) - \frac{1}{C_2 R_2} q_2(t)$$

$$y(t) = i_1(t) + i_2(t)$$

Current through $C_1 = i_1(t)$

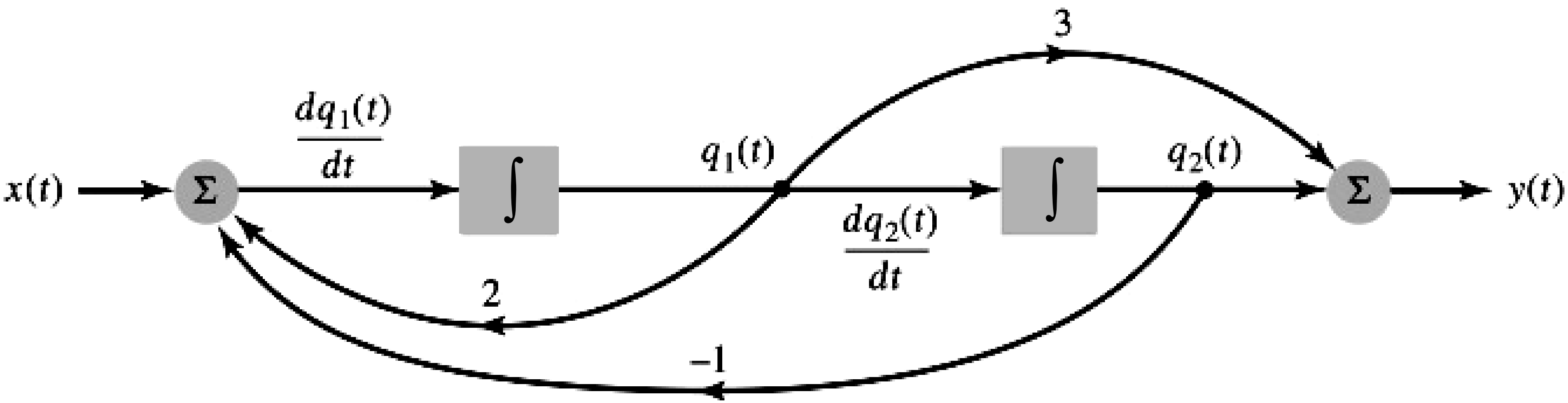
where

$$i_1(t) = C_1 \frac{d}{dt} q_1(t)$$

$$\frac{d}{dt} q_1(t) = -\left(\frac{1}{C_1 R_1} + \frac{1}{C_2 R_2}\right) q_1(t) + \frac{1}{C_1 R_2} q_2(t) + \frac{1}{C_1 R_1} x(t)$$

$$\mathbf{A} = \begin{bmatrix} -\left(\frac{1}{C_1 R_1} + \frac{1}{C_1 R_2}\right) & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{C_1 R_1} \\ 0 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -\frac{1}{R_1} & 0 \end{bmatrix}, \quad \textbf{and} \quad D = \frac{1}{R_1}$$

13. State-Variable Descriptions of LTI Systems



1. State equation:

$$\frac{d}{dt}q_1(t) = 2q_1(t) - q_2(t) + x(t)$$

$$\frac{d}{dt}q_2(t) = q_1(t)$$

2. Output equation:

$$y(t) = 3q_1(t) + q_2(t)$$

3. State-variable description:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{c} = \begin{bmatrix} 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

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Transformations of the State

- **State-variables are not unique.**
- **Different state-variable descriptions may be obtained by transforming the state variables.**
  - The new state-variables are a weighted sum of the original ones.
  - This changes the form of **A,b,c, and *D, but does not*** change the I/O characteristics of the system.

Transformations of the State

1. Original state-variable description:

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{b}x$$
$$y = \mathbf{c}\mathbf{q} + Dx$$

T = state-transformation matrix

2. Transformation: **q' = Tq** ⇒ **q = T^{-1} q'**

3. New state-variable description:

$$\dot{\mathbf{q}}' = \mathbf{T}\dot{\mathbf{q}}$$

1) State equation: $\dot{\mathbf{q}}' = \mathbf{T}\mathbf{A}\mathbf{q} + \mathbf{T}\mathbf{b}x.$

⇒ $\dot{\mathbf{q}}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{q}' + \mathbf{T}\mathbf{b}x.$

q = T^{-1} q'

2) Output equation: $y = \mathbf{c}\mathbf{T}^{-1}\mathbf{q} + Dx.$

3) If we set

$$\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}, \quad \mathbf{b}' = \mathbf{T}\mathbf{b}, \quad \mathbf{c}' = \mathbf{c}\mathbf{T}^{-1}, \quad \text{and} \quad D' = D$$

then

$\dot{\mathbf{q}}' = \mathbf{A}'\mathbf{q} + \mathbf{b}'x$ **and** $y = \mathbf{c}'\mathbf{q} + D'x$

Remarks

- Introduction
- Convolution Sum
- Convolution Sum Evaluation Procedure
- Convolution Integral
- Convolution Integral Evaluation Procedure
- Interconnection of LTI Systemss
- Relations between LTI System Properties and the Impulse Response
- Step Response

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