Chap 6 Structures for Discrete-Time Systems

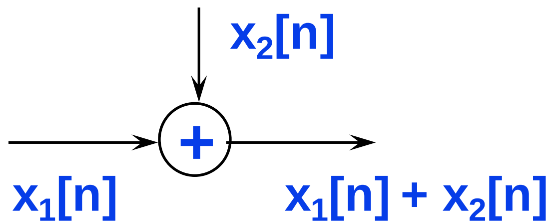
- Introduction
- Block Diagram Representation of LCCDE
- Signal Flow Graph of LCCDE
- Basic Structures for IIR Systems
- Transposed Forms
- Basic Network Structures for FIR Systems
- Lattice Structures
- Overview of Finite-Precision Numerical Effects
- Effects of Coefficient Quantization
- Effects of Roundoff Noise in Digital Filters
- Zero-Input Limit Cycles in Fixed-Point Realizations of IIR Digital Filters

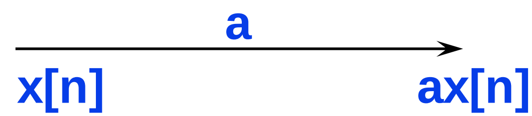
1. Introduction

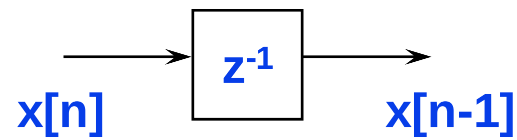
- Structures <==> Implementation
  - Interconnection of additions, multiplications, delay.
- Approaches
  - A combination of algebraic manipulations and manipulations of block diagram representations.
  - Derive equivalent equivalent structures.
- Pondering Questions
  - Finite Impulse Response
  - Infinite Impulse Response
  - Numerical Problems in Implementation

2. Block Diagram Representation of LCCDE

- Notations
  - Addition
  - Multiplication
  - Delay

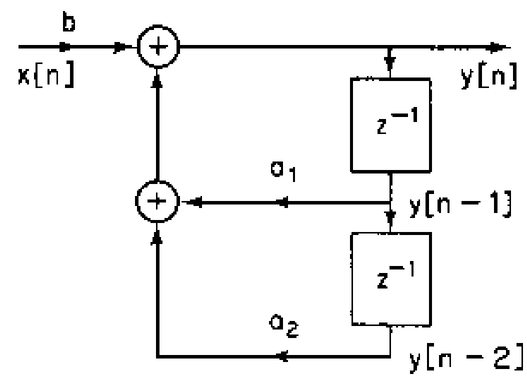






$$H(z) = \frac{b}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

- Example
  - $y[n] = a_1 y[n-1] + a_2 y[n-2] + bx[n]$



$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

2. Block Diagram Representation of LCCDE (c.1)

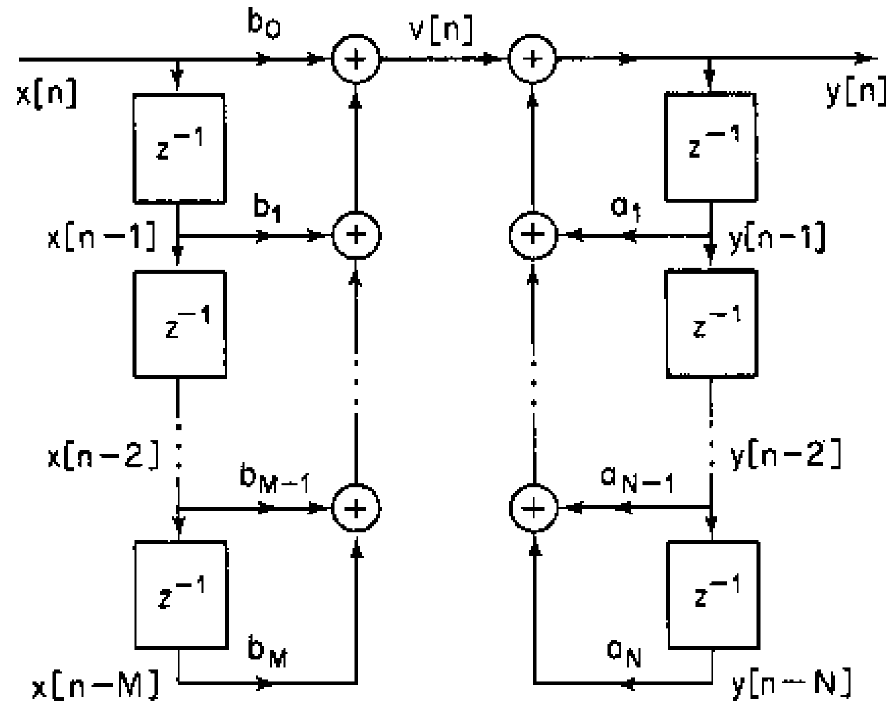
- Block Diagram 1
  - Difference Equations

$$v[n]=\sum_{k=0}^{M} b_k x[n-k]$$

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]+v[n]$$

  - Transfer Function

$$H(z)=H_2[z]H_1[z]=\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$



Direct Form I

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2. Block Diagram Representation of LCCDE (c.2)

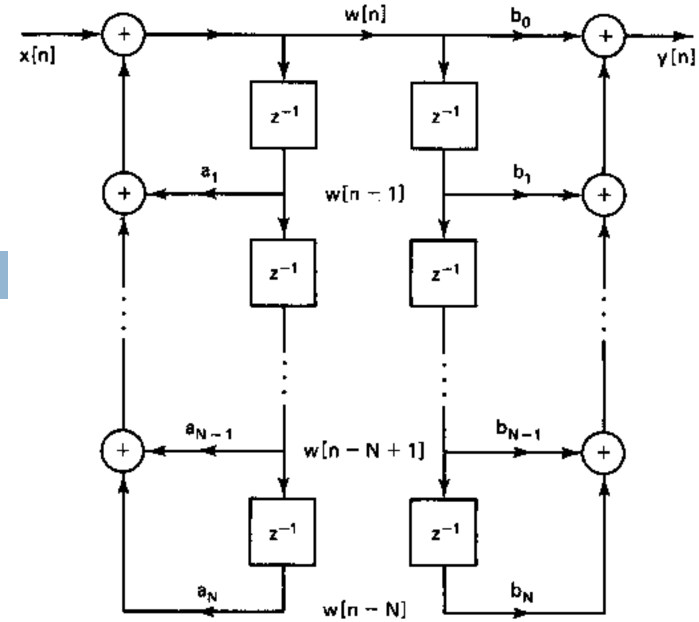
- Block Diagram 2
  - Transfer Function

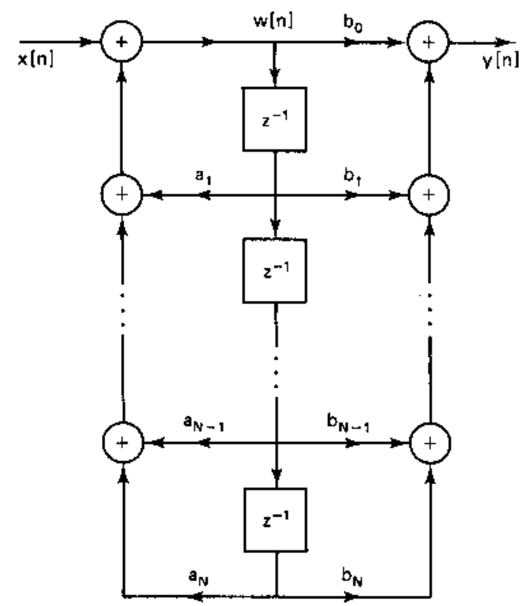
$$H(z) = H_1[z]H_2[z] = \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)$$

$$w[n] = \sum_{k=0}^{M} b_k w[n-k]$$

  - Equations

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

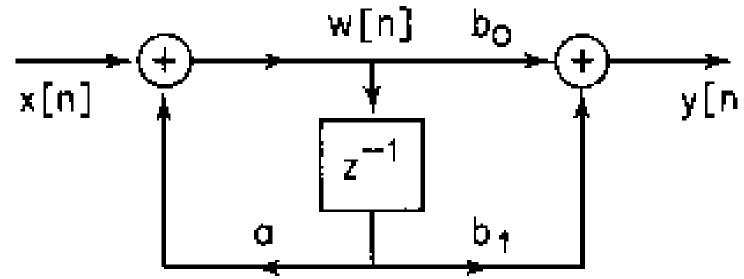


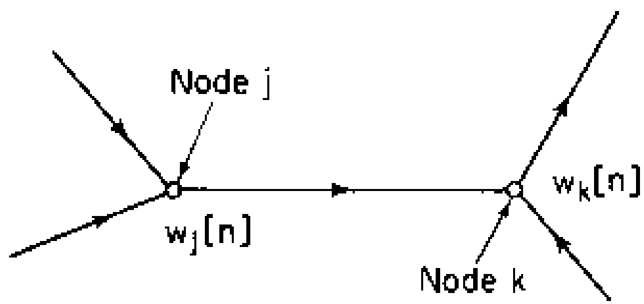


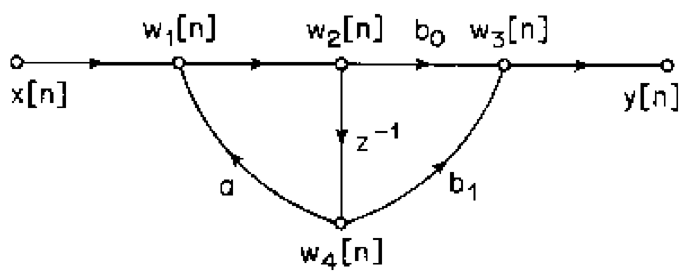
Direct Form II or Canonic Direct Form

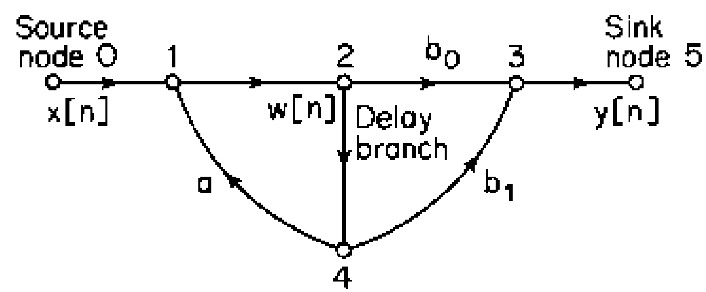
3. Signal Flow Graph of LCCDE

- Signal Flow Graph
  - A network of directed branches that at nodes.
  - Source nodes
  - Sink nodes
  - branch gains









$$w_2[n] = a w_2[n-1] + x[n]$$

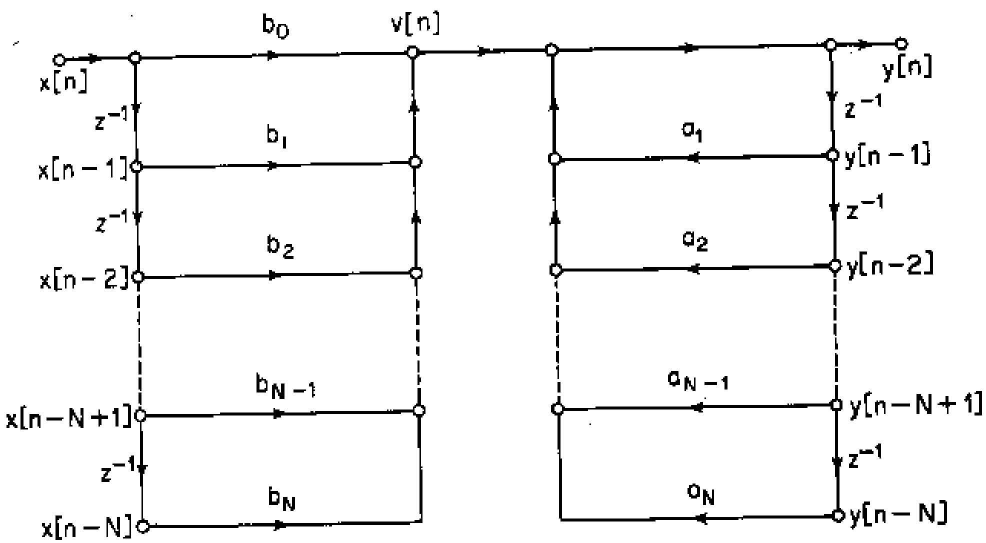
$$y[n] = b_0 w_2[n] + b_1 w_2[n-1]$$

4. Basic Structures for IIR Systems

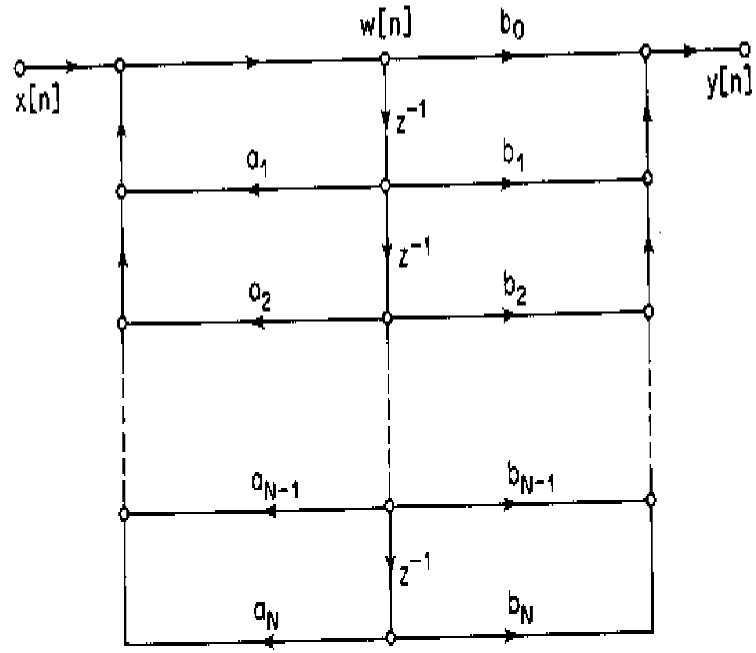
Signal Flow Graphs

- *Direct Form I*
- *Direct Form II*

Direct Form I



Direct Form II



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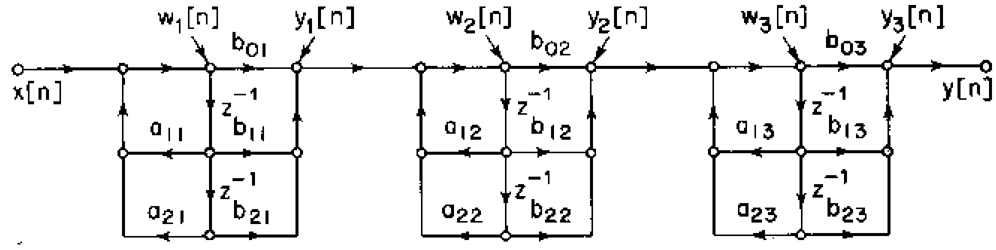
4. Basic Structures for IIR Systems (c.1)

- Cascade Form

$$H(z)=\prod_{k=1}^{N_s}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1-a_{1k}z^{-1}+a_{2k}z^{-2}}$$

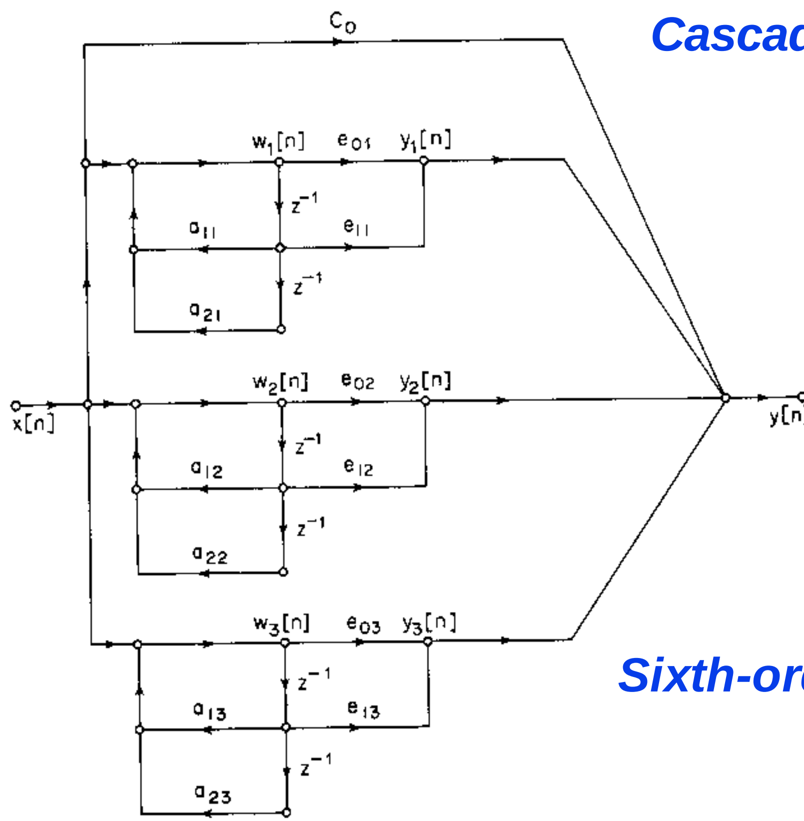


Cascade Structures

- Parallel Form

$$H(z)=\sum_{k=0}^{N_p}C_k z^{-k}+\sum_{k=1}^{N_1}\frac{A_k}{1-c_k z^{-1}}$$

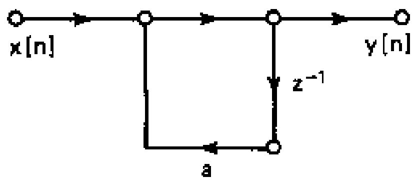
$$+\sum_{k=1}^{N_1}\frac{B_k(1-e_k z^{-1})}{(1-d_k z^{-1})(1-d_k^{*}z^{-1})}$$

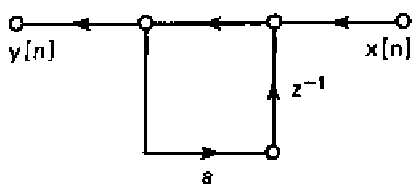


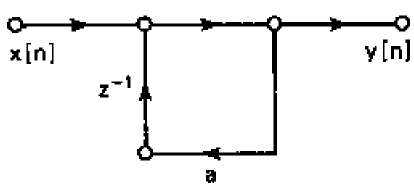
Sixth-order system

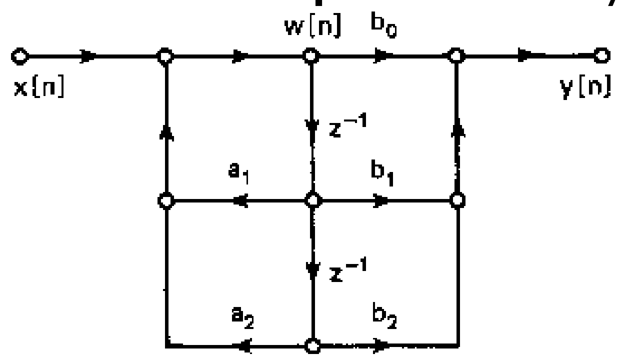
5. Transposed Forms

- Flow Graph Reversal ==> Equivalent I/O Relation

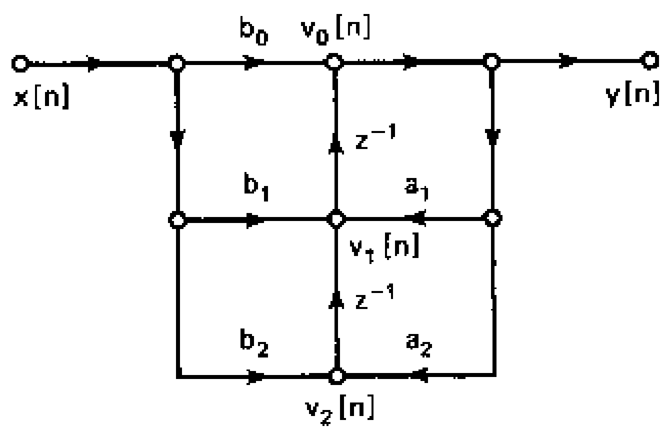








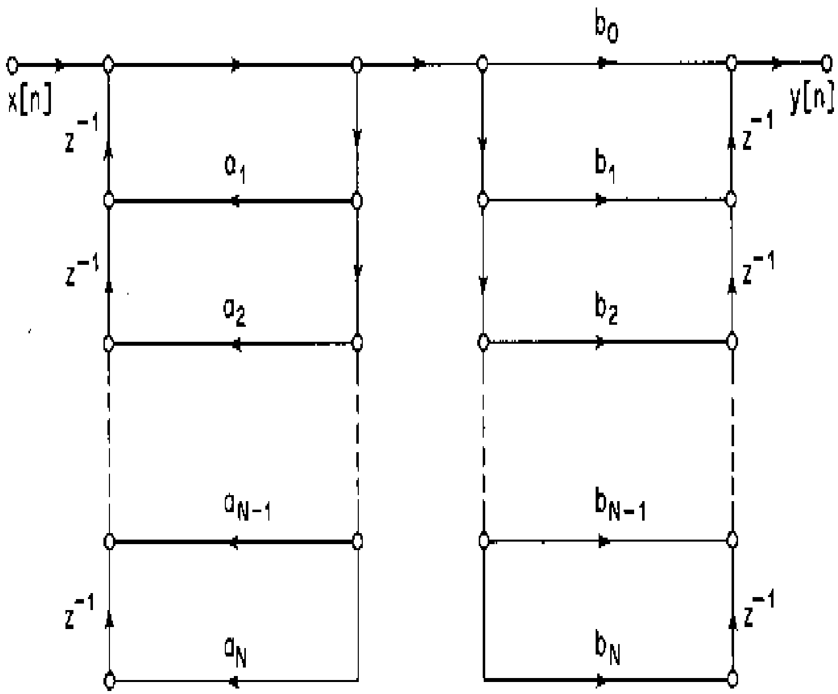
$$w[n] = a_1 w[n-1] + a_2 w[n-2] + x[n],$$
$$y[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2].$$



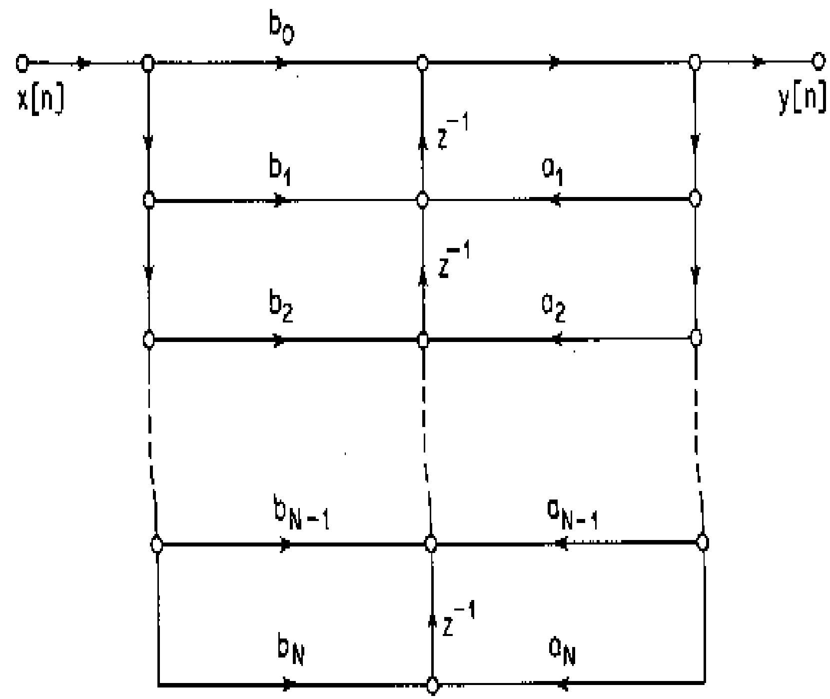
$$v_0[n] = b_0 x[n] + v_1[n-1],$$
$$y[n] = v_0[n],$$
$$v_1[n] = a_1 y[n] + b_1 x[n] + v_2[n-1],$$
$$v_2[n] = a_2 y[n] + b_2 x[n].$$

5. Transposed Forms (c.1)

- Transposed of Direct Form I



- ***Transposed of Direct Form II***



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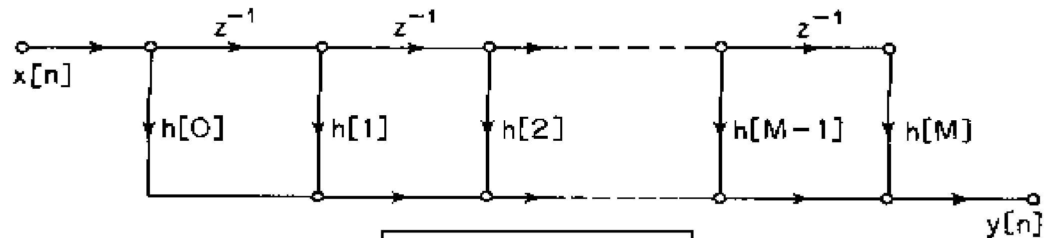
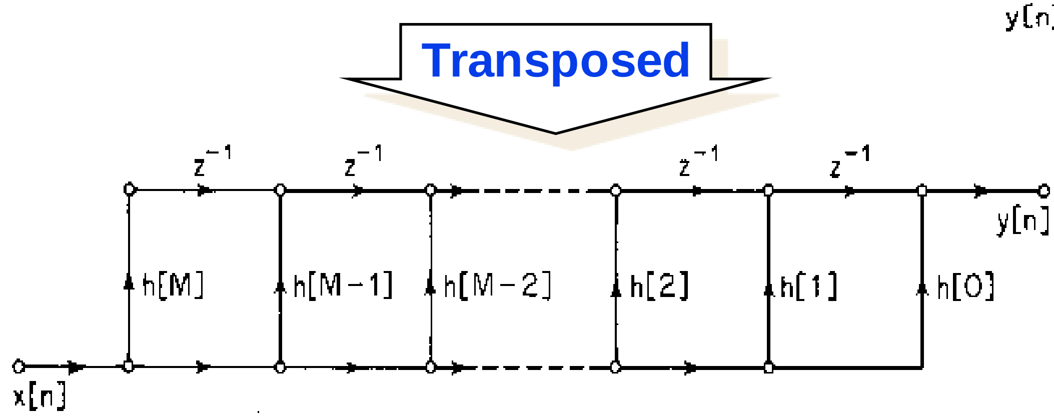
6. Basic Network Structures for FIR Systems

- Direct Form

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]$$



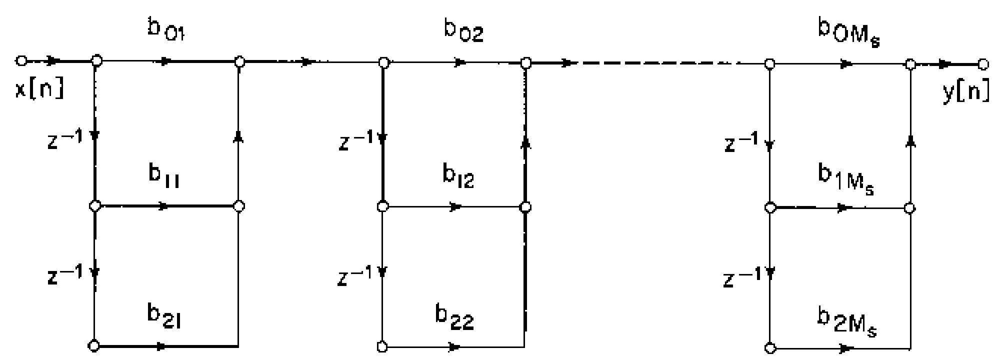


  - Derived from the direct f

- Cascade Form

$$H(z)=\sum_{k=1}^{M} h[n]z^{-n}$$

$$=\prod_{k=1}^{M_s}\left(b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}\right),\quad M_s=\lfloor (M+1)/2 \rfloor$$



6. Basic Network Structures for FIR Systems

- Linear Phase FIR Structure
  - M even for Type I and III systems

$$h[M-n]=h[n]$$

or

$$h[M-n]=-h[n]$$

  - Consider the following form

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{M} h[k]x[n-k] \\
&= \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=M/2+1}^{M} h[k]x[n-k] \\
&= \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=0}^{M/2-1} h[M-k]x[n-M+k]
\end{aligned}
$$

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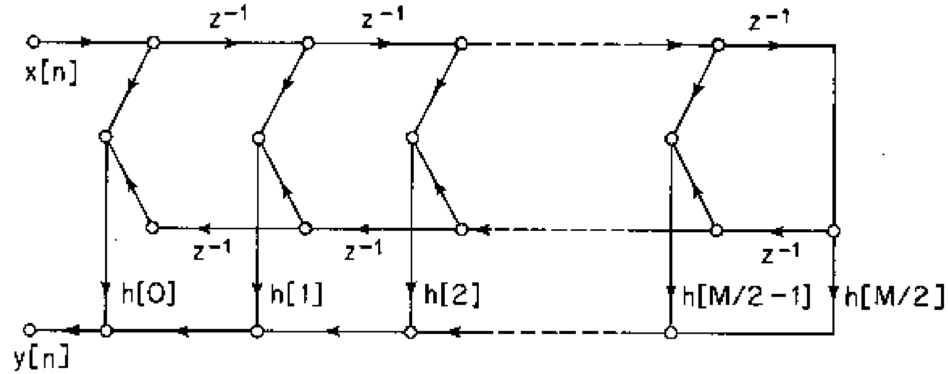
6. Basic Network Structures



- Type I Systems (M is even)

$$y[n]=\left\{\sum_{k=0}^{M/2-1} h[k](x[n-k]+x[n-M+k])\right\}+h[M/2]x[n-M/2]$$

- Type III Systems (M is even)

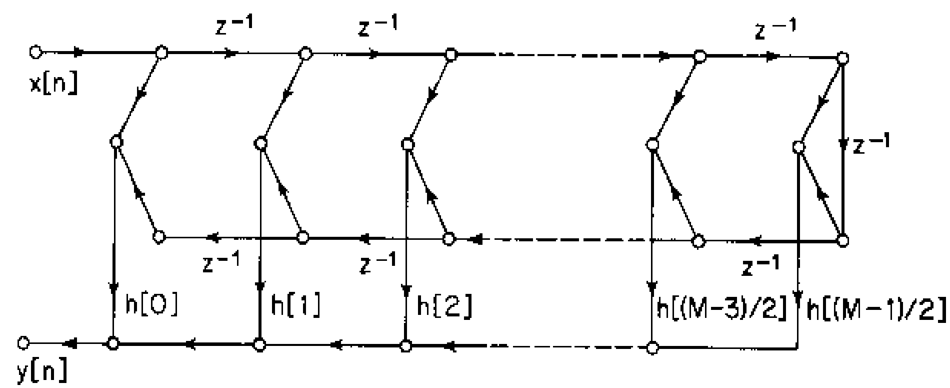
$$y[n]=\left\{\sum_{k=0}^{M/2-1} h[k](x[n-k]-x[n-M+k])\right\}+h[M/2]x[n-M/2]$$

- Type II Systems (M is odd)

$$y[n]=\left\{\sum_{k=0}^{M/2-1} h[k](x[n-k]+x[n-M+k])\right\}$$

- Type IV Systems (M is odd)

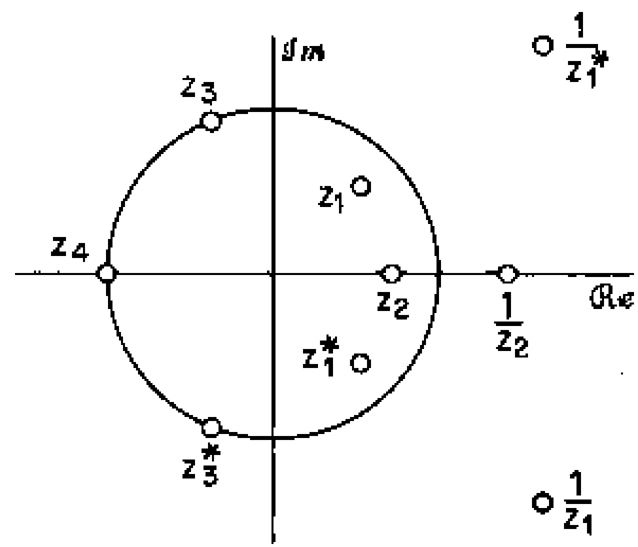
$$y[n]=\left\{\sum_{k=0}^{M/2-1} h[k](x[n-k]-x[n-M+k])\right\}$$



6. Basic Network Structures for FIR Systems (c.2)

- The zeros of a linear phase system
  - Zeros occurs in mirror-image pairs.
    - Ex. z_0 is a zero and $1/z_0$ also a zero.
  - Real zeros not on the unit circle occur in reciprocal pairs.
    - Grouped into a pair of two
  - Complex zeros occurs in group of four
- Ex.

$$H(z) = h[0](1+z^{-1})(1+az^{-1}+z^{-2})$$
$$\times(1+bz^{-1}+z^{-2})(1+cz^{-1}+dz^{-2}+cz^{-3}+z^{-4})$$



where $a = (z_2 + 1/z_2), b = 2\mathrm{Re}(z_3),$

$$c = -2\mathrm{Re}(z_1 + 1/z_1), d = 2 + |z_1 + 1/z_1|^2$$

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7. Lattice Structures

- FIR Lattice
  - An alternative of the filter structures
  - The difference equations

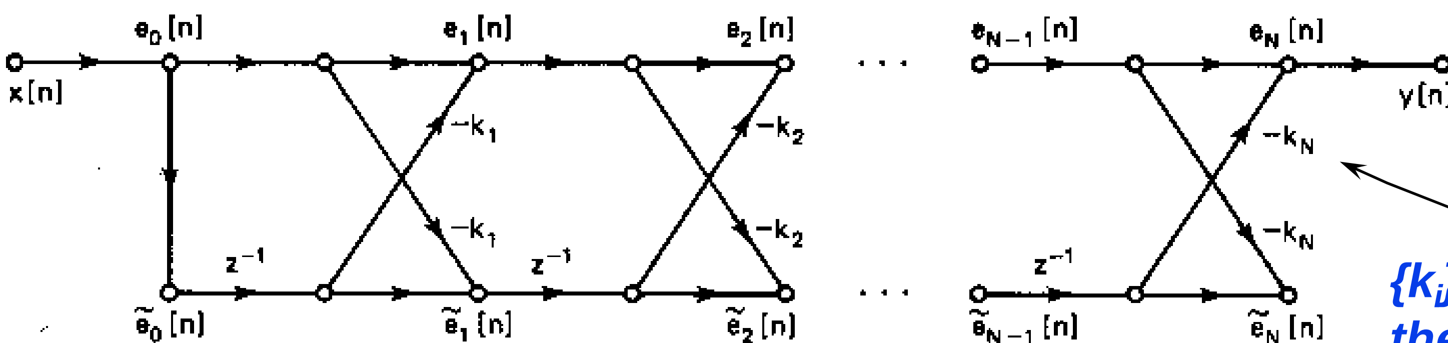
$$e_0[n] = \tilde{e}_0[n] = x[n]$$

$$e_i[n] = e_{i-1}[n] - k_i\tilde{e}_{i-1}[n-1], \quad i = 1,2,\ldots,N$$

$$\tilde{e}_i[n] = -k_i e_{i-1}[n] + \tilde{e}_{i-1}[n-1], \quad i = 1,2,\ldots,N$$

$$y[n] = e_N[n]$$

$h[0] = 1$
$h[N] = -k_N$

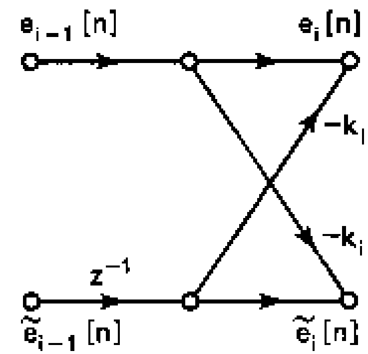


$\{k_i\}$ is the k-parameters

7. Lattice Structures (c.1)

- Relation with the impulse reponse
  - Define the impulse response

$$h[n]=\begin{cases}1, & \text{for } n=0\\ -a_n, & \text{for } n=1,2,\ldots,N.\\ 0, & \text{otherwise}\end{cases}$$



$$A_i(z)=\frac{E_i(z)}{E_0(z)}=\left[1-\sum_{m=1}^{i}a_m^{(i)}z^{-m}\right] \quad \text{and} \quad \tilde{A}_i(z)=\frac{\tilde{E}_i(z)}{\tilde{E}_0(z)}$$

  - Define

$$A_0(z)=\tilde{A}_0(z)=1;\;\; X(z)=E_0(z)=\tilde{E}_0(z);\;\; A(z)=A_N(z)$$

$$E_0(z)=\tilde{E}_0(z)=X(z)$$

$$E_i(z)=E_{i-1}(z)-k_i z^{-1}\tilde{E}_{i-1}(z), \quad i=1,2,\ldots,N$$

$$\tilde{E}_i(z)=-k_i E_{i-1}(z)+z^{-1}\tilde{E}_{i-1}(z), \quad i=1,2,\ldots,N$$

$$Y(z)=E_N(z)$$

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7. Lattice Structures (c.2)

- Recurrence Formula
  - The coefficient of z^{-m} is

$$a_i^{(i)} = k_i\,,$$
$$a_m^{(i)} = a_m^{(i-1)} - k_i a_{i-m}^{(i-1)}, \qquad m = 1,2,\ldots,(i-1)$$

$$a_m = a_m^{(N)}, \quad m = 1,2,\ldots,N.$$

Reflection coefficients or PARCOR coefficients

- The final set of coefficients is

$$a_m^{(N)} = a_m\,, \quad m = 1,2,\ldots,N.$$

  - The recursion is repeated for I=1, 2, ..., N, and the final set of coefficients for $A(z)=A_N(z)$ is

$$k_i = a_i^{(i)}\,,$$
$$a_m^{(i-1)} = \frac{a_m^{(i)} + k_i a_{i-m}^{(i)}}{1-k_i^2}, \qquad m = 1,2,\ldots,(i-1)$$

PSP LAB

7. Lattice Structures (c.3)

- Example

$$A(z) = (1-0.8jz^{-1})(1+0.8jz^{-1})(1-0.9z^{-1})$$
$$= 1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}$$

$$k_3 = a_3^{(3)} = 0.576,$$

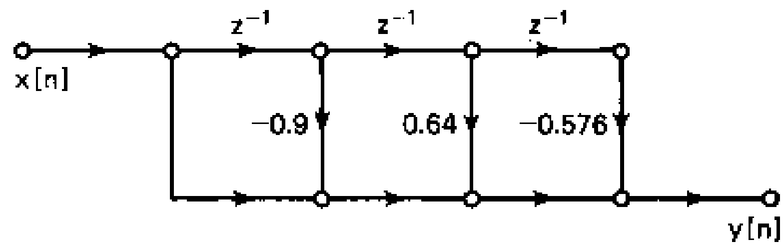
$$a_1^{(2)} = \frac{a_1^{(3)} + k_3 a_2^{(3)}}{1 - k_3^2} = 0.79518245,$$

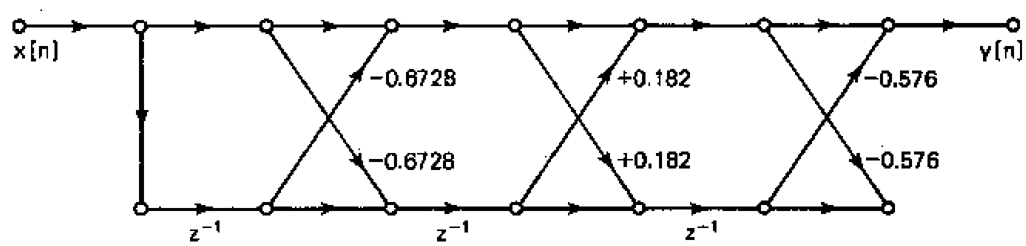
$$a_2^{(2)} = \frac{a_2^{(3)} + k_3 a_1^{(3)}}{1 - k_3^2} = -0.18197491,$$

$$k_2 = a_2^{(2)} = -0.18197491,$$

$$a_1^{(1)} = \frac{a_1^{(2)} + k_2 a_1^{(2)}}{1 - k_2^2} = 0.67275747,$$

$$k_1 = a_1^{(1)} = 0.67275747.$$





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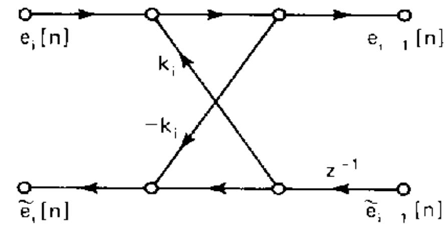
7. Lattice Structures-- All-Pole Lattice

- A Lattice System
  - H(z) = 1/A(z)

Stability Test from *ki* ?

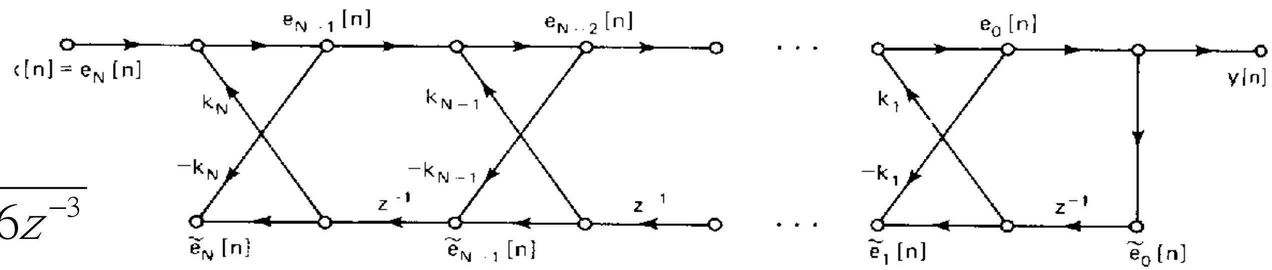


$$e_N[n] = x[n],$$

$$e_{i-1}[n] = e_i[n] + k_i\tilde{e}_{i-1}[n-1], \qquad i = N, (N-1), \ldots, 1,$$

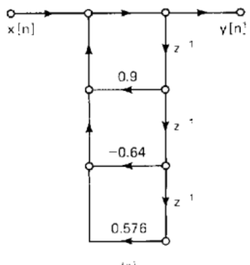
$$\tilde{e}_i[n] = -k_i e_{i-1}[n] + \tilde{e}_{i-1}[n-1], \qquad i = N, (N-1), \ldots, 1,$$

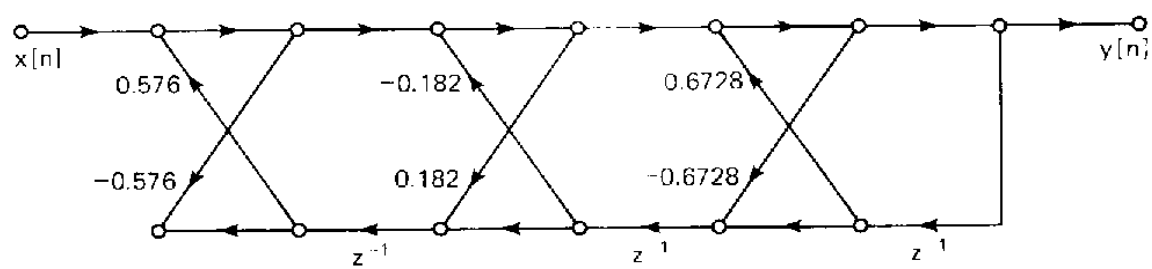
$$y[n] = e_0[n] = \tilde{e}_0[n].$$



$$H(z) = \frac{1}{1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}}$$

- Example





7. Lattice Structures-- Normalized Lattice

Equations of a Cell

$$e_N[n] = x[n],$$

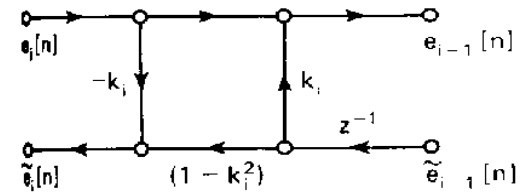
$$e_{i-1}[n] = e_i[n] + k_i\tilde{e}_{i-1}[n-1],$$

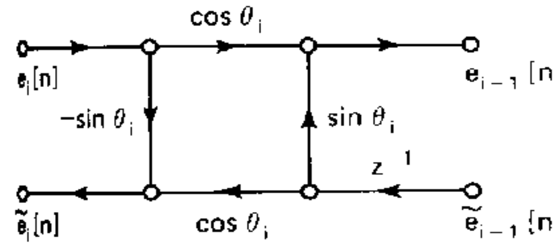
$$\tilde{e}_i[n] = -k_i e_i[n] + (1-k_i^2)\tilde{e}_{i-1}[n-1],$$

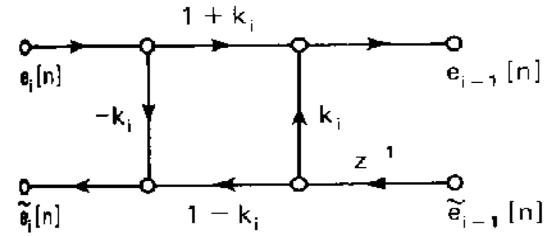
$$y[n] = e_0[n] = \tilde{e}_0[n].$$

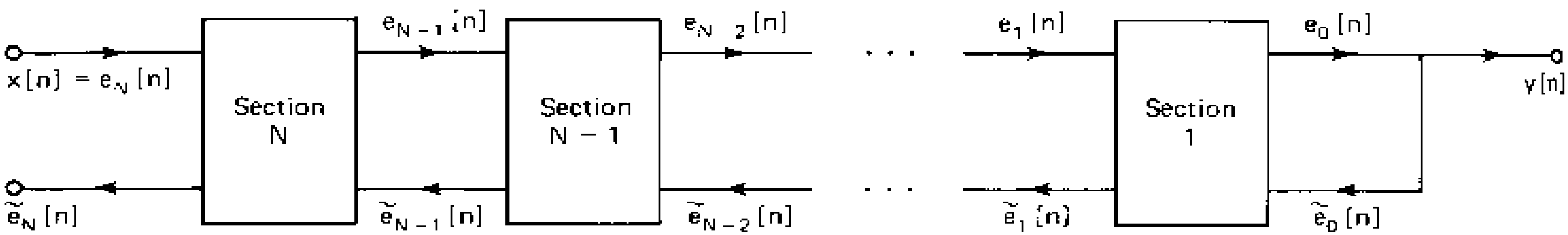
$$\sin\theta_i = k_i$$

$$\cos\theta_i = \sqrt{1-k_i^2}$$









7. Lattice Structures-- Lattice Systems with Poles and Zeros

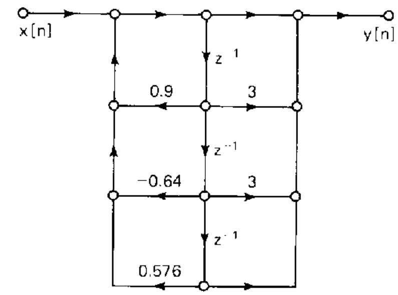
- Equations

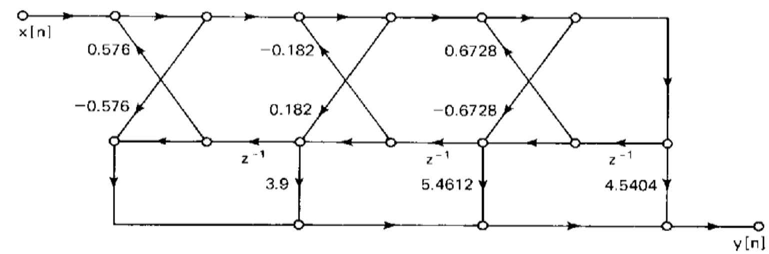
$$\tilde{E}_i(z) = z^{-i} A_i(z^{-1}) \tilde{E}_0(z) = z^{-i} A_i(z^{-1}) Y(z)$$

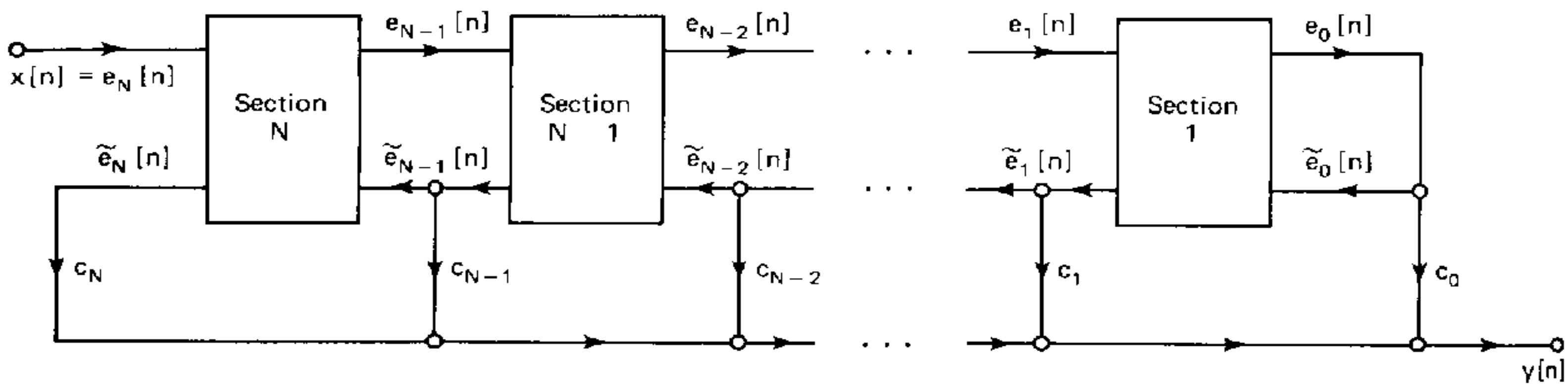
$$\tilde{H}_i(z) = \tilde{E}_i(z) / X(z) = \frac{z^{-i} A_i(z^{-1})}{A(z)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{N} \frac{c_i z^{-i} A_i(z^{-1})}{A(z)} = \frac{B(z)}{A(z)}$$

$$B(z) = \sum_{m=0}^{N} b_m z^{-m} \qquad b_m = c_m - \sum_{i=m+1}^{N} c_i a_{i-m}^{(i)}$$



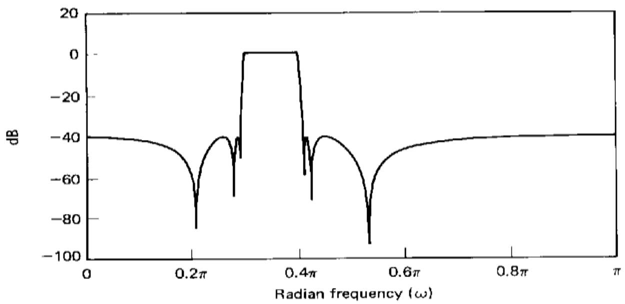




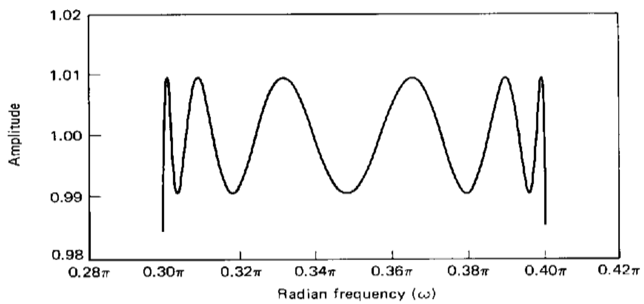
7. Effects of Coefficient Quantization

- Specification

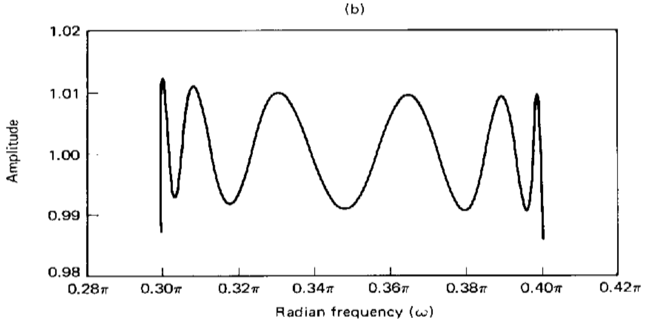
$$0.99 < |H(e^{j\omega})| \le 1.01, \qquad 0.3\pi < \omega \le 0.4\pi,$$
$$|H(e^{j\omega})| \le 0.01 \text{ (i.e., } -40 \text{ dB)}, \qquad \omega \le 0.29\pi,$$
$$|H(e^{j\omega})| \le 0.01 \text{ (i.e., } -40 \text{ dB)}, \qquad 0.41\pi \le \omega \le \pi.$$

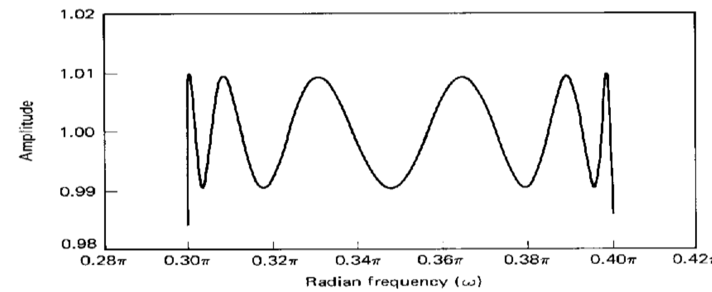


(a)

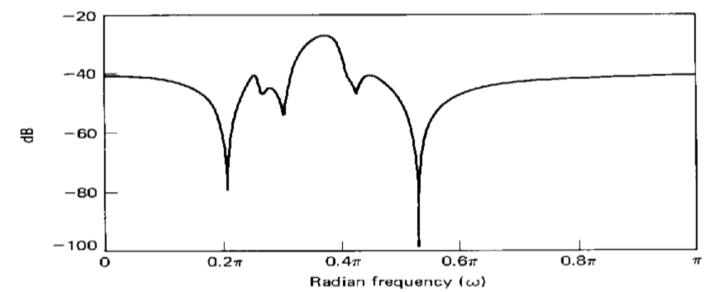


(b)

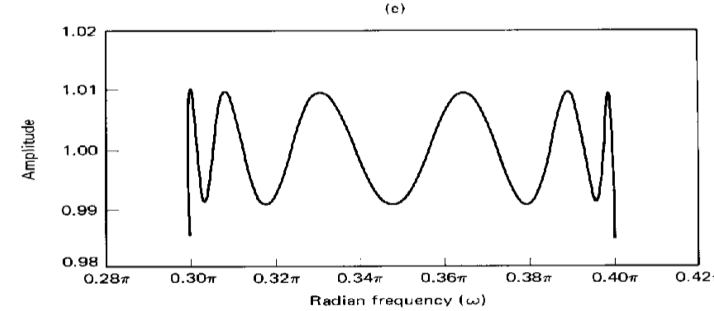




(d)



(c)



(f)

7. Effects of Coefficient Quantization

- Analysis from Poles and Zeros
  - The denominator

$$A(z) = 1 - \sum_{k=1}^{N} a_k z^{-k} = \prod_{j=1}^{N} (1 - z_j z^{-1})$$

  - The error of ith pole

$$\Delta z_i = \sum_{k=1}^{N} \frac{\partial z_i}{\partial a_k} \Delta a_k , \quad i = 1, 2, \ldots, N.$$

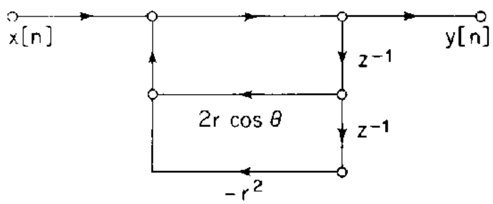
  - Since that $\left(\frac{\partial A(z)}{\partial z_i}\right)_{z=z_i} \frac{\partial z_i}{\partial a_k} = \left(\frac{\partial A(z)}{\partial a_k}\right)_{z=z_i}$.

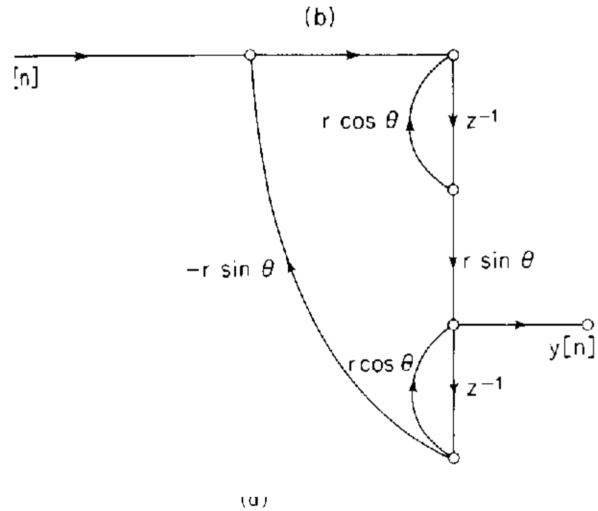
  - It follows that

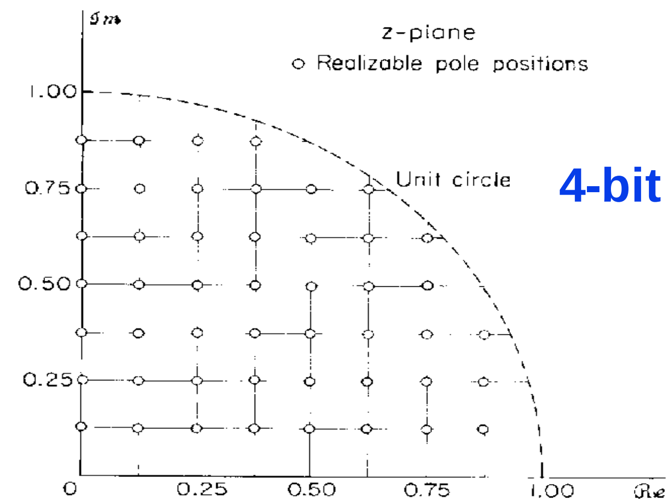
$$\frac{\partial z_i}{\partial a_k} = \frac{z_i^{N-k}}{\prod_{j=1, j\neq i}^{N} (z_i - z_j)}$$

- **Narrow bandwidth Filters**
- **Implementation through modular structures like cascade or parallel forms**

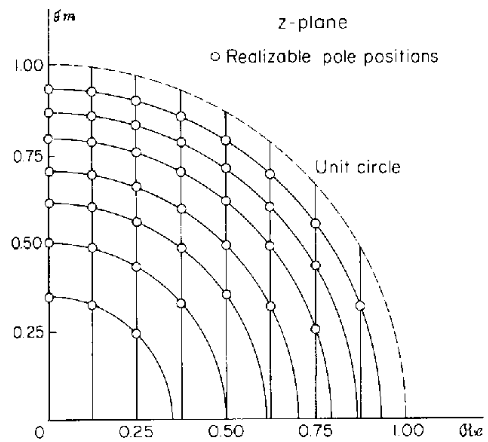
7. Effects of Coefficient Quantization



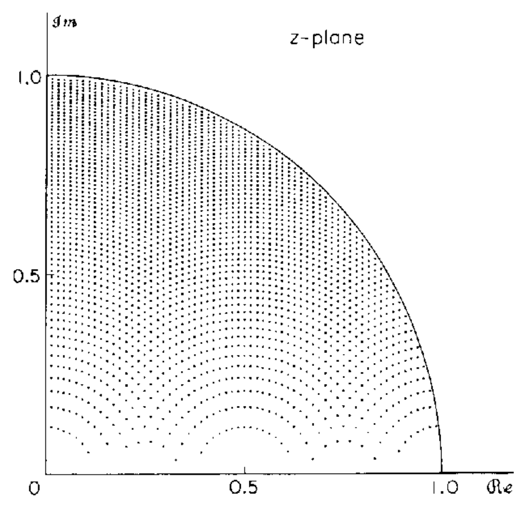




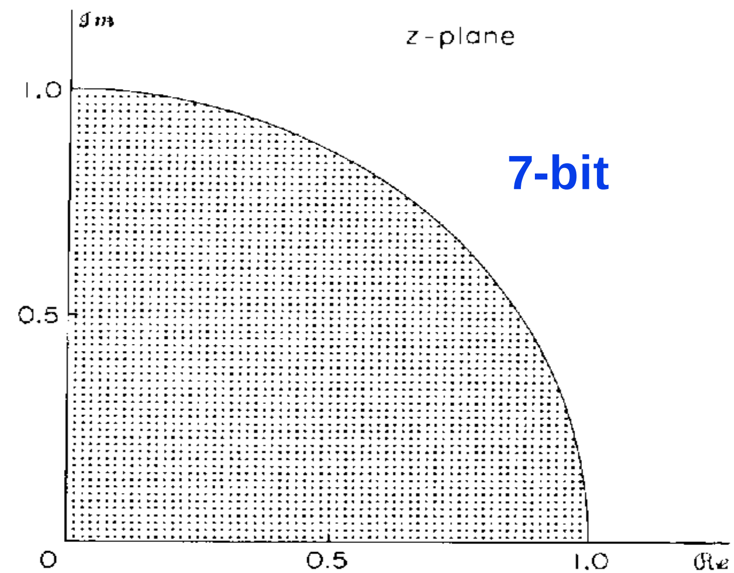
4-bit



4-bit



7-bit



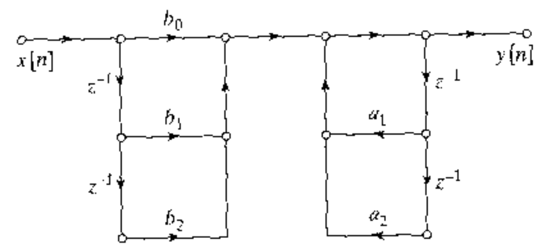
7-bit

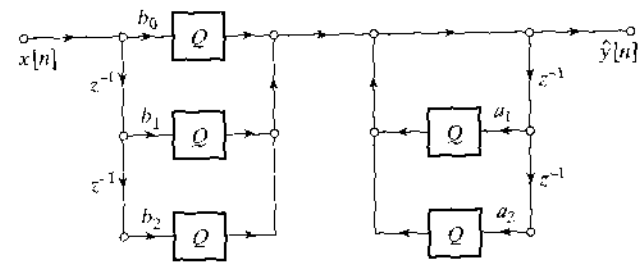
8. Effects of Round-off Noise in Digital Filters

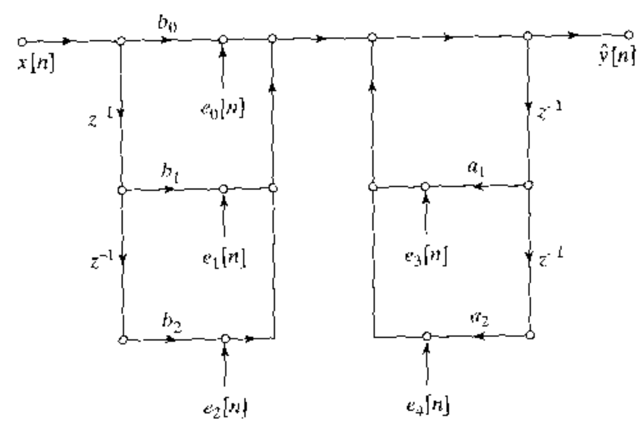
- IIR Structure
  - (B+1) bit fixed point

$$y[n] = \sum_{i=1}^{N} a_k y[n-i] + \sum_{k=0}^{M} b_k x[n-k]$$

$$\hat{y}[n] = \sum_{i=1}^{N} Q[a_k \hat{y}[n-i]] + \sum_{k=0}^{M} Q[b_k x[n-k]]$$







- Assumption on Noise
  - e[n] is wide-sense stationary white-noise.
  - A uniform distribution of amplitude.
  - Noise is uncorrelated with the input.

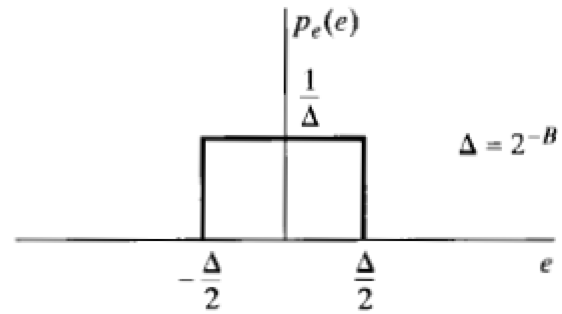
Quantization

For $(B+1)$-bit quantization, we showed in Section 6.6 that, for rounding,

$$-\frac{1}{2}2^{-B} < e[n] \le \frac{1}{2}2^{-B},$$

$$m_e = 0,$$

$$\sigma_e^2 = \frac{2^{-2B}}{12}.$$



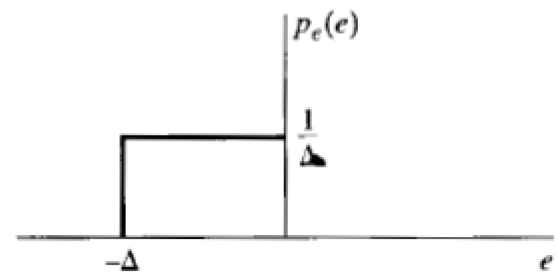
(a)

for two's-complement truncation,

$$-2^{-B} < e[n] \le 0.$$

$$m_e = -\frac{2^{-B}}{2},$$

$$\sigma_e^2 = \frac{2^{-2B}}{12}.$$



(b)

PSP LAB

8. Effects of Round-off Noise in Digital Filters (c.1)

- Quantization Errors
  - For (B+1)-bit quantization

$$-\frac{1}{2}2^{-B} < e[n] \le \frac{1}{2}2^{-B}$$

  - For 2's complement

$$-2^{-B} < e[n] \le 0$$

- Autocorrelation sequence

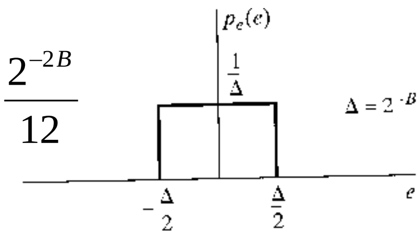
$$\phi_{ee}[n] = \sigma_e^2\,\delta[n] + m_e^2.$$

- Total noise

$$e[n] = e_0[n] + e_1[n] + e_2[n] + e_3[n] + e_4[n].$$

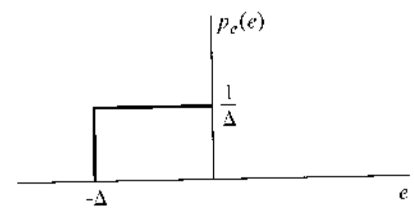
$$\sigma_e^2 = \sigma_{e_0}^2 + \sigma_{e_1}^2 + \sigma_{e_2}^2 + \sigma_{e_3}^2 + \sigma_{e_4}^2 = 5 \cdot \frac{2^{-2B}}{12},$$

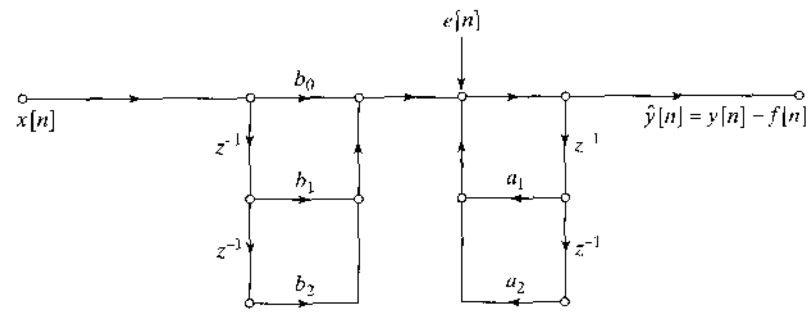
$$m_e = 0; \sigma_e^2 = \frac{2^{-2B}}{12}$$



$$m_e = -\frac{2^{-B}}{2},$$

$$\sigma_e^2 = \frac{2^{-2B}}{12}.$$





8. Effects of Round-off Noise in Digital Filters (c.2)

- General Form

$$\sigma_e^2 = (M+1+N)\frac{2^{-2B}}{12}.$$

$$f[n] = \sum_{k=1}^{N} a_k f[n-k] + e[n];$$

$$m_f = m_e \sum_{n=-\infty}^{\infty} h_{ef}[n] = m_e H_{ef}(e^{j0}).$$

$$P_{ff}(\omega) = \Phi_{ff}(e^{j\omega}) = \sigma_e^2 |H_{ef}(e^{j\omega})|^2.$$

$$\sigma_f^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{ff}(\omega)d\omega = \sigma_e^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{ef}(e^{j\omega})|^2 d\omega.$$

$$\sigma_f^2 = \sigma_e^2 \sum_{n=-\infty}^{\infty} \left|h_{ef}[n]\right|^2$$

Parserval's relationship.

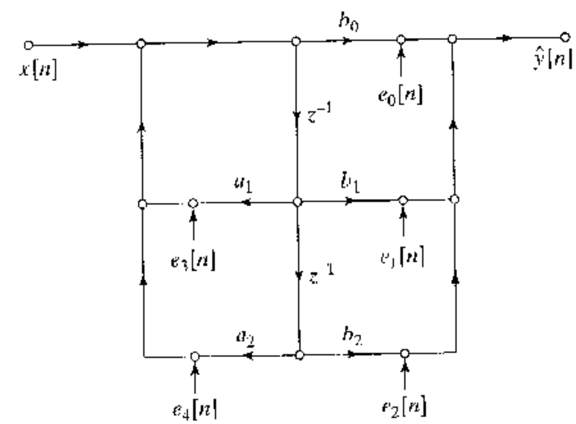
PSP LAB

8. Effects of Round-off Noise in Digital Filters (c.3)

- Noise Variance

$$P_{ff}(\omega) = N\frac{2^{-2B}}{12}\left|H(e^{j\omega})\right|^2 + (M+1)\frac{2^{-2B}}{12}$$

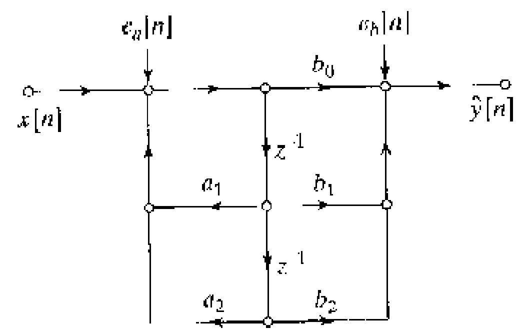
$$\boxed{\begin{aligned}\sigma_f^2 &= N\frac{2^{-2B}}{12}\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2 d\omega + (M+1)\frac{2^{-2B}}{12}\\ &= N\frac{2^{-2B}}{12}\sum_{n=-\infty}^{\infty}|h[n]|^2 + (M+1)\frac{2^{-2B}}{12}.\end{aligned}}$$



$$\hat{y}[n] = Q\left[\sum_{k=1}^{N} a_k \hat{y}[n-k] + \sum_{k=0}^{M} b_k x[n-k]\right];$$

$$\hat{w}[n] = Q\left[\sum_{k=1}^{N} a_k \hat{w}[n-k] + x[n]\right]$$

$$\hat{y}[n] = Q\left[\sum_{k=0}^{M} b_k \hat{w}[n-k]\right].$$



8. Effects of Round-off Noise in Digital Filters – Scaling in Fixed-Point Implementation

- The overflow concerns

$$|w_k[n]| = \left| \sum_{m=-\infty}^{\infty} x[n-m]h_k[m] \right|. \qquad |w_k[n]| < 1$$

- Sufficient condition for

$$|w_k[n]| \le x_{\max} \sum_{m=-\infty}^{\infty} |h_k[m]| \qquad x_{\max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k[m]|}$$

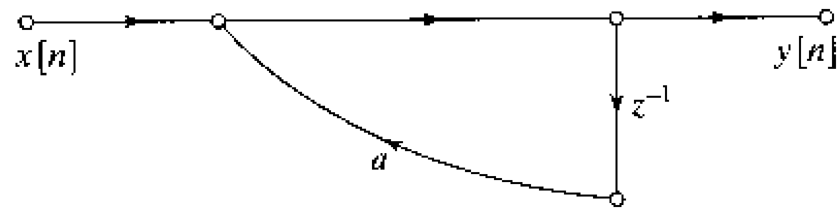
- Hence

$$s\,x_{\max} < \frac{1}{\max_k \left[\sum_{m=-\infty}^{\infty} |h_k[m]| \right]}.$$

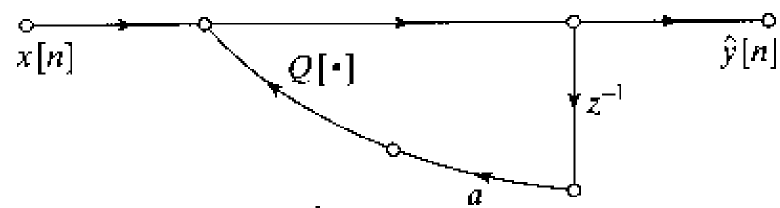
9. Zero-Input Limit Cycles in Fixed-Point

- Due to Round-off and Truncation

$$y[n] = ay[n-1] + x[n], \qquad |a| < 1.$$

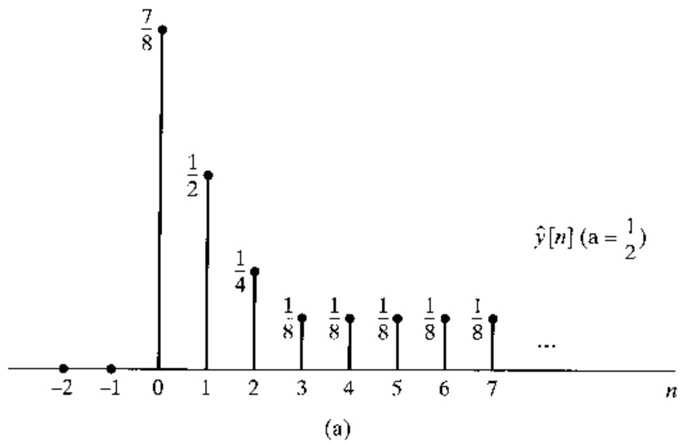


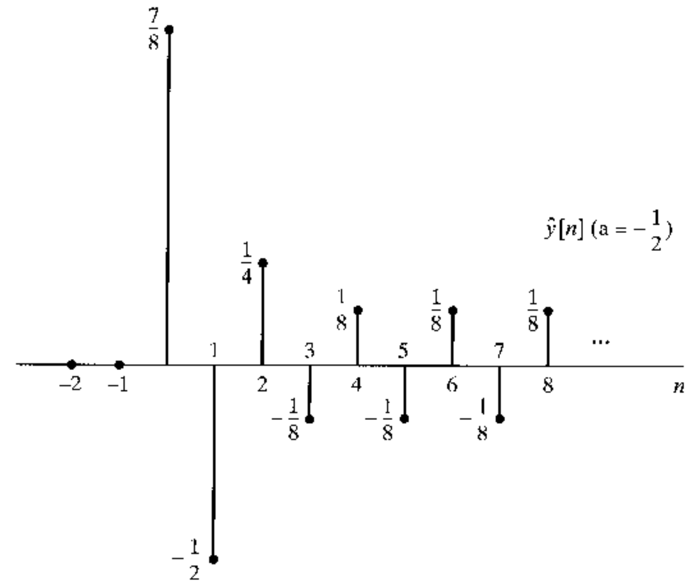
$$\hat{y}[n] = Q[a\,\hat{y}[n-1]] + x[n],$$



where $Q[\cdot]$ represents the rounding operation. Let us assume that $a = 1/2 = 0_\diamond 100$ and that the input is $x[n] = (7/8)\delta[n] = (0_\diamond 111)\delta[n]$. Using Eq. (6.120), we see that for $n = 0$, $\hat{y}[0] = 7/8 = 0_\diamond 111$. To obtain $\hat{y}[1]$, we multiply $\hat{y}[0]$ by a, obtaining the result $\hat{a}\, y[0] = 0_\diamond 011100$, a 7-bit number that must be rounded to 4 bits. This number, 7/16, is exactly halfway between the two 4-bit quantization levels 4/8 and 3/8. If we choose always to round upward in such cases, then $0_\diamond 011100$ rounded to 4 bits is $0_\diamond 100 = 1/2$.

9. Zero-Input Limit Cycles in Fixed-Point





Homeworks

- 6.23, 6.24d, 6.25, 6.40, 6.42, 6.45
- 2.82, 2.85, 2.90