The z-Transform

- Introduction
- z-Transform
- Properties of the Region of Convergence for the z-Transform
- Inverse z-Transform
- z-Transform Properties
- Unilateral z-Transform
- Solving the Difference Equations
- Zero-Input Response
- Transfer Function representation
- Summary

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0. Introduction

- Role in Discrete-Time Systems
  - z-Transform is the discrete-time counterpart of the Laplace transform.
- Response of Discrete-Time Systems
  - If the system

    $2y[n] + 3y[n-1] + y[n-2] = u[n] + u[n-1] - u[n-2] \qquad \text{for } n = 0, 1, 2$
  - The response of the system is excited by an input u[n] and some initial conditions.
  - The difference equations are basically algebraic equations, their solutions can be obtained by direct substitution.
  - The solution however is not in closed form and is difficult to develop general properties of the system.
  - A number of design techniques have been developed in the z-Transform domain.

1. The z-Transform

- Positive and Negative Time Sequence
  - A discrete-time signal x[n] = xa[nT], where n is an integer ranging (-*<n<*), is called a positive-time sequence if x[n]=0 for n < 0; it is called a negative-time sequence if x[n] = 0 for n > 0.
  - We mainly consider the positive-time sequences.
- z-Transform Pair
  - The z-transform is defined as

$$X(z) \equiv Z[x[n]] \equiv \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

    - where z is a complex variable, called the z-transform variable.

Difference with the Fourier Transform

- Example
  - x[n] = { 1, 2, 5, 7, 0, 1}; x[n] = $(1/2)^n$ u{n}

1. The z-Transform (c.1)

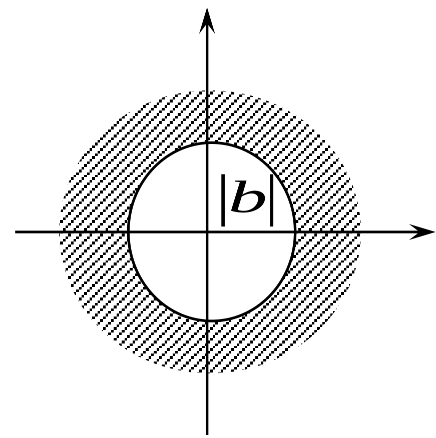
- Example
  - f[n] = b^n for all positive integer k and b is a real or complex number.

$$F(z)=\sum_{n=0}^{\infty} f[n]z^{-n}=\sum_{n=0}^{\infty} b^n z^{-n}=\sum_{n=0}^{\infty}(bz^{-1})^n$$

  - If $|bz^{-1}|<1$, then the infinity power series converges and

$$F(z)=\frac{1}{1-bz^{-1}}=\frac{z}{z-b}$$

  - The region $|b|<|z|$ is called the region of convergence.



- Unit Step Sequence
  - The unit sequence is defined as

$$q[n]=\begin{cases}1 & for\ n=0,1,2,\\ 0 & for\ n<0\end{cases}$$

  - The z-Transform is

$$Q(z)=\sum_{n=0}^{\infty} z^{-n}=\frac{1}{1-z^{-1}}$$

- Exponential Sequence
  - f[n] = e^{anT}

$$F(z)=\sum_{n=0}^{\infty} e^{anT}z^{-n}=\frac{1}{1-e^{aT}z^{-1}}$$

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1. The z-Transform (c.2)

- Region of Convergence
  - For any given sequence, the set of values of z for which the z-transform converges is called the region of convergence.
- Viewpoints
  - The representation of the complex variable z

$$z = re^{j\omega}$$

  - Consider the z-transform

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

  - Convergent Condition

$$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$$

ROC includes the unit circle ==> Fourier Transform converges

Convergence of the z-Transform ==> The z-transform and its derivatives must be continuous function of z.

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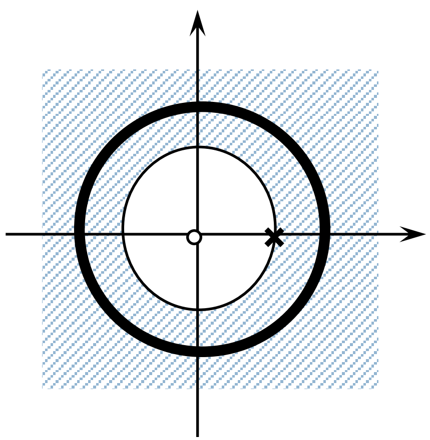
1. The z-Transform (c.3)

- Rational Function

$$X(z) = \frac{P(z)}{Q(z)}$$

- Ex.

$$x[n] = a^n u[n] \qquad x[n] = -a^n u[-n-1]$$



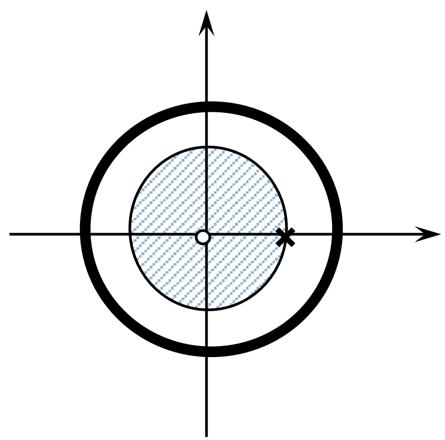


TABLE 4.1 SOME COMMON *z*-TRANSFORM PAIRS

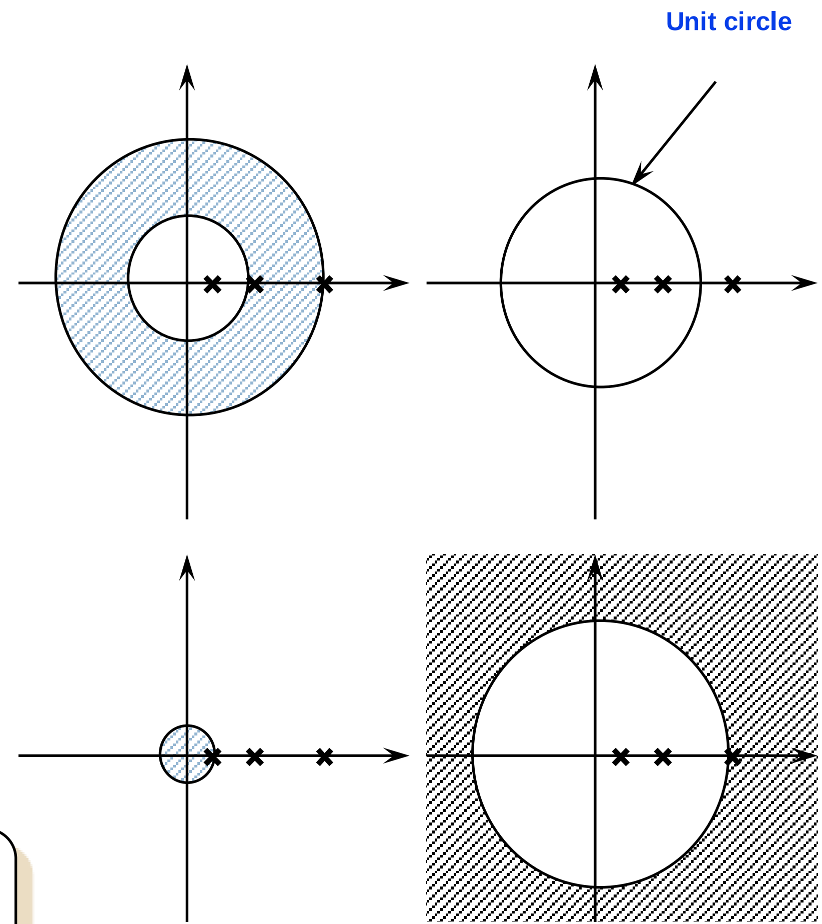
Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

2. Properties of the Region of Convergence for the z-Transform

- Properties
  - The ROC is a ring or disk in the z-plane centered at the origin, i.e., $0 \le r_R < |z| < r_L \le \infty$
  - The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform of x[n] includes the unit circle.
  - The ROC cannot contain any poles.
  - If x[n] is a finite-duration sequence, i.e. a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 \le \infty$, then the ROC is the entire z-plane except possibly z=0 or z= ∞.
  - If x[n] is a right-sided sequence, i.e. a sequence that is zero for $n<N_1<\infty$, the ROC extends outward from the outermost finite pole in X(z) to z= ∞ .
  - If x[n] is a left-sided sequence, i.e., a sequence that is zero for $n>N_2>-\infty$, the ROC extends inward from the innermost (smallest magnitude) nonzero pole in X(z) to (and possibly including) z=0.
  - A two-sided sequence is an infinite-duration sequence that is neither right-sided nor left-sided. If x[n] is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole, and, consistent with property 3, not containing any poles.
  - The ROC must be a connected region.

2. Properties of the Region of Convergence for the z-Transform

- Example
  - ROC is a Ring
  - ROC is the interior of a circle
  - ROC is the outerior of a circle



No common ROC case ?

3. The Inverse z-Transform

- Methods
  - Direct Division
  - Partial Fraction Expansion

$$F(z) = \frac{-2z^4 + 5z^3 + z^2 - 6z + 3}{z^2 - z - 2}$$

- Direct Division
  - $F(z) = -2z^2 + 3z + 3z^{-2} + 3z^{-3} + 9z^{-4}$

$$\begin{array}{r|l} & -2z^2 + 3z \qquad + 3z^{-2} + 3z^{-3} + 9z^{-4} \\ \hline z^2 - z - 2 & -2z^4 + 5z^3 + z^2 - 6z + 3 \\ & -2z^4 + 2z^3 + 4z^2 \\ \hline & 3z^3 - 3z^2 - 6z + 3 \\ & 3z^3 - 3z^2 - 6z \\ \hline & +3 \end{array}$$

$f[k] = \{-2, 3, \underset{\uparrow}{0}, 0, 3, 3, 9, \ldots\}$

- Ex. $3/(z^2 - z - 2)$

Inverse z-transform by Power Series Expansion

Example 3.11 Inverse Transform by Power Series Expansion

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Using the power series expansion for $\log(1 + x)$, with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}.$$

Therefore,

$$x[n] = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$

3. The Inverse z-Transform (c.1)

- Partial Fraction Expansion and Table Lookup

$$X(z)=\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad X(z)=\frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

- If M<N and the poles are all first order

$$X(z)=\frac{b_0}{a_0}\sum_{k=1}^{N}\frac{A_k}{(1-d_k z^{-1})} \qquad A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

- If M>= N and the poles are all first order, the complete partial fraction expression can be

$$X(z)=\sum_{r=0}^{M-N}B_r z^{-r}+\sum_{k=1}^{N}\frac{A_k}{(1-d_k z^{-1})} \qquad A_k=(1-d_k z^{-1})X(z)\Big|_{z=d_k}$$

- If X(z) has multiple-order poles and M>=N

$$X(z)=\sum_{r=0}^{M-N}B_r z^{-r}+\sum_{k=1,k\neq i}^{N}\frac{A_k}{(1-d_k z^{-1})}+\sum_{m=1}^{s}\frac{C_m}{(1-d_i z^{-1})^m}$$

$$C_m=\frac{1}{(s-m)!(-d_i)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}[(1-d_i w)^s X(w^{-1})]\right\}_{w=d_i^{-1}}$$

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3. The Inverse z-Transform (c.2)

- Examples

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

- ROC: $|z|>1$

- ROC: $|z|<\frac{1}{2}$

- ROC: $\frac{1}{2}<|z|<1$

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4. z-Transform Properties

The Initial-Value Theorem

Let F(z) be the z-transform of f(n) a positive-time sequence f[n], n = 0, 1, 2, ..., and let F(z) be a proper rational function, then

$$f(0) = \lim_{z \to \infty} F(z)$$

This follows from F(z) = f[0] + f[1]z^{-1} + f[2]z^{-2}+ ...

SOME z-TRANSFORM PROPERTIES

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{Jm}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Initial-value theorem:

$x[n] = 0, \quad n < 0 \qquad \lim_{z \to \infty} X(z) = x[0]$

4. z-Transform Properties (c.1)

- The Final-Value Theorem
  - Let F(z) be the z-transform of f[n], n=0, 1, 2, ... and let F(z) be a proper rational and let F(z) be a proper rational function. If every pole of (z-1)F(z) has a magnitude smaller than 1, then f[k] approaches a constant and

$$\lim_{k\to\infty} f[k] = \lim_{z\to 1}(z-1)F(z)$$

- Examples

$$\frac{3z}{(z-1)(z+1)};\ \frac{z+20}{(z+0.9)^{10}};\ \frac{z+1}{3(z^2-1)(z+0.9)};\ \frac{(2z+1)(z-10)}{z(z+2)}$$

consider f[k] = 2^k

- proof

$$Z[f[k]] = F(z) = \lim_{N\to\infty}\sum_{k=0}^{N} f[k]z^{-k}$$

$$Z\{f[k+1]\} = z[F(z) - f(0)] = \lim_{N\to\infty}\sum_{k=0}^{N} f[k+1]z^{-k}$$

$$(z-1)F(z) - zf[0] = \lim_{N\to\infty}\sum_{k=0}^{N}[f[k+1]z^{-k} - f[k]z^{-k}]$$

$$As\ z->1, the\ right-hand\ side\ reduces\ to\ (f[N+1]-f[0]).$$

$$\lim_{z\to 1}(z-1)F(z) - f[0] = \lim_{z\to 1} f[N+1] - f(0)$$

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5. The Unilateral z-Transform

- **Definition**

$$X(z) \equiv Z[x[n]] \equiv \sum_{n=0}^{\infty} x[n] z^{-n}$$

- **Time Delay**

$$x[n] \Longleftrightarrow X(z)$$

$$x[n-k] \Longleftrightarrow z^{-k} X(z) + \sum_{n=1}^{k} x[-n] z^{-k+n}$$

$$x[n+k] \Longleftrightarrow z^{i} \left[X(z) - \sum_{n=0}^{k-1} x[n] z^{-n} \right]$$

6. Solving the Difference Equations

- Goals

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  - Solving CC difference equations using Z transform.

- Example

  $2y[n] + 3y[n-1] + y[n-2] = u[n] + u[n-1] - u[n-2]$

  where $u(n) = s(n)$, $y(-1) = -1$ and $y(-2) = 1$.

- Exercise

  Find the response of

  $y[n+1] - 2y[n] = u[n]$ and $y[n+1] - 2y[n] = u[n+1]$ $y[-1]=1$

  and $u[n] = 1$, for $n = 0, 1, 2, \ldots$

7. Zero-Input Response-- Characteristic Polynomial

- Consider the zero-input response

$$Y_{zi}(z) = \frac{-3y[-1] - y[-2] - y[-1]z^{-1}}{2 + 3z^{-1} + z^{-2}} = \frac{(-3y[-1] - y[-2])z^{2} - y[-1]z}{2z^{2} + 3z + 1}$$

  - The denominator of Y_{zi}, is called the characteristic polynomial.
  - The roots of Y_{zi}, is called the modes of the system.
- Why the name, "mode" ?
  - The zero-input response of the system excited by any initial conditions can always be expressed as

$$Y_{zi}(z) = \frac{(-3y[-1] - y[-2])z^{2} - y[-1]z}{2z^{2} + 3z + 1} = \frac{k_1 z}{z + 0.5} + \frac{k_2 z}{z + 1}$$

  - The zero-input response is for t>=0

$$y_{zi}(k) = k_1(-0.5)^k + k_2(-1)^k$$

The form of the zero-input response excited by any initial conditions is completely determined by the modes of the system

  - The zero-input response is always a linear combination of the two time functions $(-0.5)^k$ and $(-1)^k$.

7. Zero-State Response-- Transfer Function

- Consider
  $2y[k] + 3y[k-1] + y[k-2] = u[k] + u[k-1] - u[k-2]$
  - If all initial conditions are zero, we have

$$Y(z) = \frac{1 + z^{-1} - z^{-2}}{2 + 3z^{-1} + z^{-2}} U(z) = \frac{z^2 + z - 1}{2z^2 + 3z + 1} U(z)$$

- Ways to Find Transfer Functions
  - The transfer function is the z transform of the impulse response.
  - The function can be obtained from the zero-state response excited by any input, in particular, step or sinusoidal functions.
  - The function can be obtained from the difference equation description.
  - $y[n]-y[n-1]+2y[n-2]-3y[n-3]= u[n]$

The Transfer Functions

The ratio of the z transforms of the output and input with all initial conditions zero or

$$H(z) = \left. \frac{Y(z)}{U(z)} \right|_{initial\ conditions\ =\ 0}$$

The transfer function of the above example is $(z^2 + z - 1)/(2z^2 + 3z + 1)$

7.1 Poles and Zeros of Proper Transfer Functions

- For a proper rational functions

$$H(z) = \frac{N(z)}{D(z)}$$

  - where N(z) and D(z) are polynomials with real coefficients. If N(z) and D(z) have no common factors, then the roots of D(z) and N(z) are respectively the poles and zeros of the system.
    - examples $$Y(z) == \frac{3(z+4)}{2(z+0.5)(z+1)} U(z)$$
- Definition
  - A finite real or complex number λ is a pole of H(z) if the absolute value of H(λ) = ∗. It is a zero of H(z) if H(λ) = 0.
- Examples
  - Find the zero-state response of a system with transfer function, H(z) = $(z^2+z-1)/(2z^2 + 3z +1)$ excited by unit step function.

- Consider the following systems

$$H_1(z) = \frac{0.551}{(z+0.9)(z-0.8+j0.5)(z-0.8-j0.5)}$$
$$= \frac{0.551}{z^3 - 0.7z^2 - 0.55z + 0.801}$$

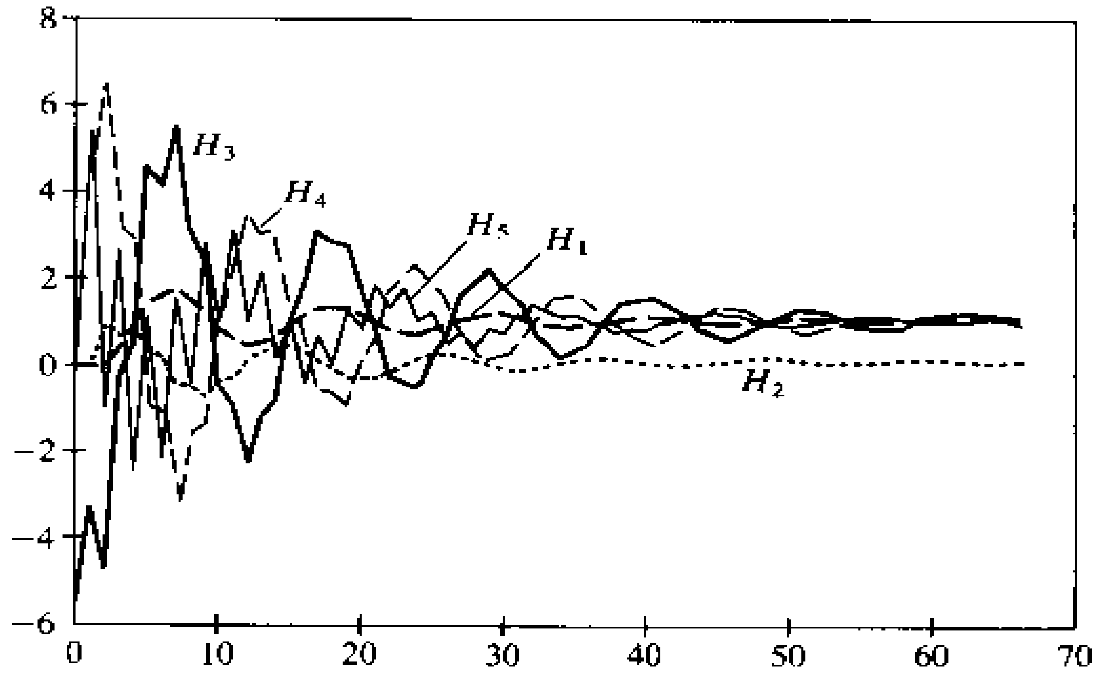
$$H_2(z) = \frac{z-1}{z^3 - 0.7z^2 - 0.55z + 0.801}$$

$$H_3(z) = \frac{5.51(z-0.9)}{z^3 - 0.7z^2 - 0.55z + 0.801}$$

$$H_4(z) = \frac{-5.51(z-1.1)}{z^3 - 0.7z^2 - 0.55z + 0.801}$$

$$H_5(z) = \frac{5.51(z^2-1.9z+1)}{z^3 - 0.7z^2 - 0.55z + 0.801}$$

Positions of zeros and poles

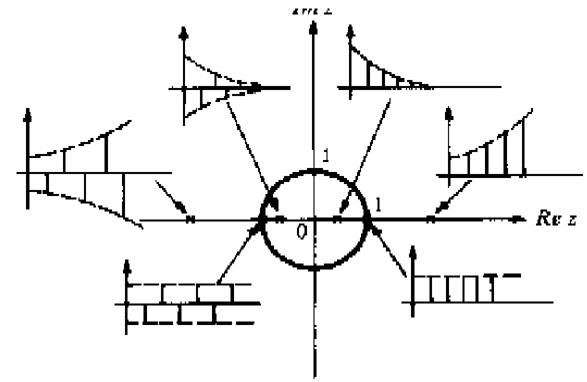


7.2 Time Responses of Modes and Poles

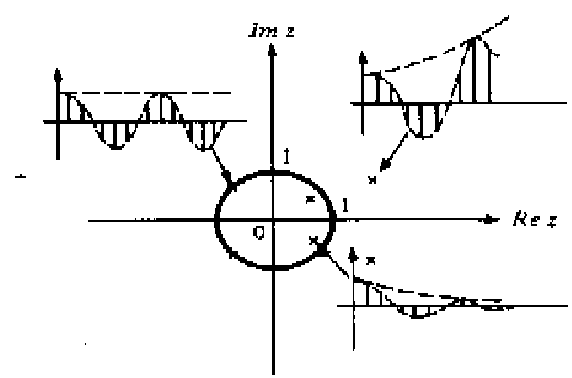
Remarks

- The zero-input response is essentially dictated by the modes; the zero-state response is essentially dictated by the poles.

- Three parts in z-plane
  - The unit circle
  - The region outside the unit circle.
  - The region inside the unit circle.



- Observations
  - The poles σ +- $j\omega$ or $re^{+-j\theta}$
  - The response $r^k \cos k\theta$ or $r^k \sin k\theta$
  - θ determines the frequency of the oscillation.
  - The highest frequency is determined by $(-\pi /T, \pi$



7.2 Time Responses of Modes and Poles(c.1)

Summary

- The time response of a pole (mode), simple or repeated, approaches zero as k -->* if and only if the pole (mode) lies inside the unit circle or its magnitude is smaller than 1.
- The time response of a pole (mode) approaches a nonzero constant as k -->* if and only if the pole is simple and located at the z=1.

Examples

$$y[k+3] - 1.6\, y[k+2] - 0.76 y[k+1] - 0.08\, y[k] = u[k+2] - 4u[k].$$

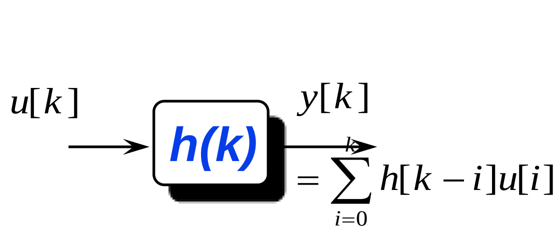
$y_{zi}[k]$, $H(z)$, $y_{zs}[k]$ as k -> *

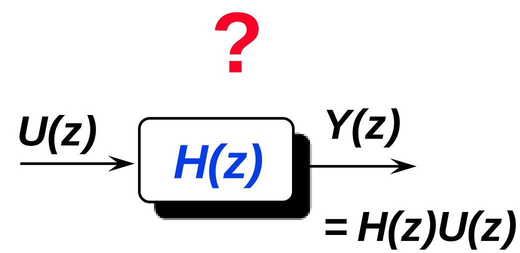
Table 5.3 *Time responses of poles and modes as* $k \to \infty$

Poles or modes	Simple ($n = 1$)	Repeated ($n = 2, 3, \ldots$)
Inside the unit circle	0	0
Outside the unit circle	$\pm\infty$	$\pm\infty$
$1/(z-1)^n$	A constant	∞
$1/[(z-e^{j\theta})(z-e^{-j\theta})]^n$	A sustained oscillation	$\pm\infty$

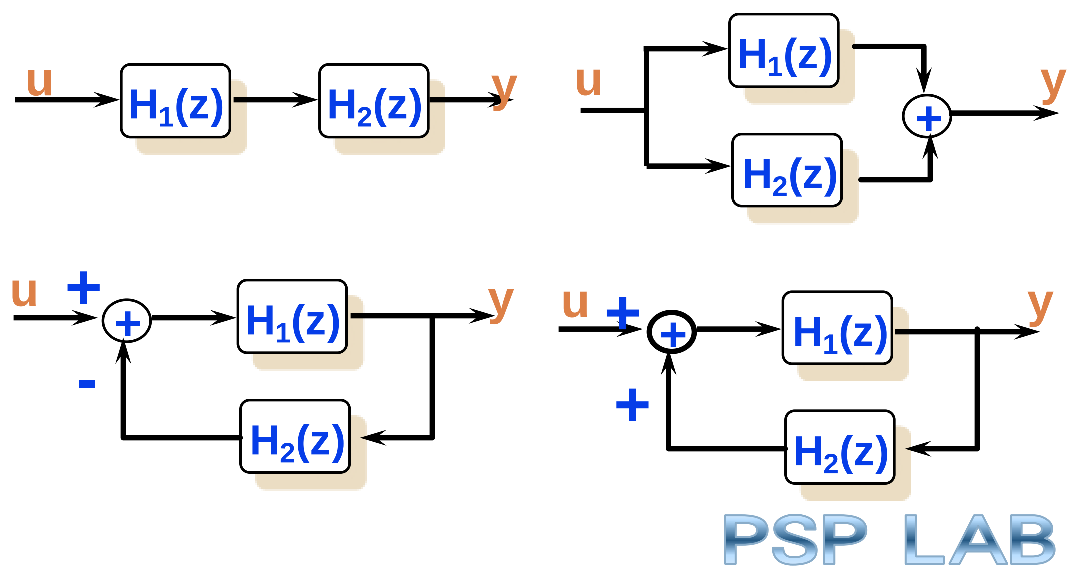
8 Transfer-Function Representation-- Complete Characterization

- System Description for LTI systems
  - Convolution
  - Difference equation
  - Transfer function.





- System Connection
  - The transfer function of connection can be easily derived from algebraic manipulation of transfer functions.



8 Transfer-Function Representation-- Complete Characterization (c.1)

Questions

- Transfer functions represent the input-output relation when initial conditions are set to zeros.
- What is the representation of the transfer function for a system

- Consider the difference equations
  - D(z) Y(z) = N(z) U (z)

  Ex. D(z) = (z+0.5)(z-1)(z-2); N(z)= z^2 - 3z +2

  $$H(z) = \frac{(z-1)(z-2)}{(z+0.5)(z-1)(z-2)} = \frac{1}{z+0.5}$$

  $y_{zs}[k] = k_1(-0.5)^k$ + (terms due to the poles of U(z))

- Missing Poles
  - If N(z) and D(z) have common factors, R(z), then the roots of R(z) are modes but not poles of the system and

    {the set of the poles} {the set of the modes}
  - The root of R(z) are called the missing poles.
- Completed Characterization
  - A system is completed characterized by its transfer function if N(z) and D(z) are coprime.

9. Summary

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- z-Transform
- Properties of the Region of Convergence for the z-Transform
- Inverse z-Transform
- z-Transform Properties
- Unilateral z-Transform
- Solving the Difference Equations
- Zero-Input Response
- Transfer Function Representation

Homeworks

- 3.22, 3.27, 3.28, 3.36, 3.37, 3.43, 3.46, 3.51