Problem 2.

Without loss of generality we can assume that the source alphabet is $\{1, 2, \cdots, M\}$. Let H_1 represent the first order entropy and let H be the entropy of the source. Define

$$G_K = -\sum_{i_1=1}^{M} \cdots \sum_{i_K=1}^{M} P(X_1 = i_1, X_2 = i_2 \cdots X_K = i_K) \log_2 P(X_1 = i_1, X_2 = i_2 \cdots X_K = i_K)$$

then by definition

$$H = \lim_{K \to \infty} \frac{1}{K} G_K$$

If the X_i are independent we can write

$$P(X_1 = i_1, X_2 = i_2 \cdots X_K = i_K) = P(X_1 = i_1) P(X_2 = i_2) \cdots P(X_K = i_K)$$

and

$$\begin{aligned} G_K &= -\sum_{i_1=1}^{M} \cdots \sum_{i_K=1}^{M} P(X_1 = i_1) P(X_2 = i_2) \cdots P(X_K = i_K) [\log_2 P(X_1 = i_1) \\ &\quad + \log_2 P(X_2 = i_2) + \cdots \log_2 P(X_K = i_K)] \\ &= -\sum_{i_2=1}^{M} P(X_2 = i_2) \cdots \sum_{i_K=1}^{M} P(X_K = i_K) \sum_{i_1=1}^{M} P(X_1 = i_1) \log_2 P(X_1 = i_1) \\ &- \sum_{i_1=1}^{M} P(X_1 = i_1) \sum_{i_3=1}^{M} P(X_3 = i_3) \cdots \sum_{i_K=1}^{M} P(X_K = i_K) \sum_{i_2=1}^{M} P(X_2 = i_2) \log_2 P(X_2 = i_2) \\ &\vdots \\ &- \sum_{i_1=1}^{M} P(X_1 = i_1) \cdots \sum_{i_{K-1}=1}^{M} P(X_{K-1} = i_{K-1}) \sum_{i_K=1}^{M} P(X_K = i_K) \log_2 P(X_K = i_K) \end{aligned}$$

The first summations in each term sum to 1 and the last summation (assuming identical distributions) are equal to H_1, and

$$G_K = K H_1$$

Therefore,

$$H = \lim_{K \to \infty} \frac{1}{K} K H_1 = H_1$$

Problem 3.

a) 2 bits.

b) 1.75 bits.

c) 1.739818 bits.

Problem 4.

$$\begin{aligned} H_Q - H_P &= -\sum_{i=1}^{m} q_i \log_2 q_i + \sum_{i=1}^{m} p_i \log_2 p_i \\ &= -q_{j-1} \log_2 q_{j-1} - q_j \log_2 q_j + p_{j-1} \log_2 p_{j-1} + p_j \log_2 p_j \end{aligned}$$

Given a function

$$f_a(x) = -x \log x - (a-x)\log(a-x)$$

we can easily show that $f_a(x)$ is maximum for $x = \frac{a}{2}$ Let

$$p_{j-1} + p_j = c$$

then

$$q_{j-1} = q_j = \frac{c}{2}$$

Then

$$\begin{aligned} H_Q - H_P &= -\frac{c}{2}\log_2\frac{c}{2} - \frac{c}{2}\log_2\frac{c}{2} + p_j \log_2 p_j + (c-p_j)\log_2(c-p_j) \quad (1) \\ &= f_c(\frac{c}{2}) - f_c(p_j) \\ &\geq 0 \end{aligned}$$

Therefore $H_Q \geq H_P$.

Problem 8.

a) Start with the list of codewords

$$\{0, 01, 11, 111\}$$

0 is a prefix for 01 generating a dangling suffix of 1. 11 is a prefix to 111 also generating a dangling suffix of 1. Augment the codeword list with the dangling suffix.

$$\{0, 01, 11, 111, 1\}$$

Now 1 is a prefix to 111 generating a dangling suffix of 11. As 11 is a codeword the code is not uniquely decodeable.

b) Start with the list of codewords {0, 01, 110, 111}. 0 is a prefix for 01, which generates a dangling suffix of 1. Augment the codeword list with the dangling suffix.

{0, 01, 110, 111, 1}.

Now, 1 is a prefix of 110 and 111 generating the dangling suffixes 10 and 11.

We augment the list and continue, which gives us {0, 01, 110, 111, 1, 10, 11}.

As 11 is a prefix of both 110 and 111, it generates dangling suffixes 0 and 1, which are both already in the list. Thus, this code is not uniquely decodable.

An example of message that cannot be decoded by this code is 01110. This message has two possible ways of decoding: 01–110 or 0–111–0.

c) Start with the list of codewords

$$\{0, 10, 110, 111\}$$

No codeword is a prefix of any other codeword. Therefore, this code is uniquely decodeable.

d) Start with the list of codewords

$$\{1, 10, 110, 111\}$$

1 is a prefix of 110 which generates a dangling suffix of 10 which is a codeword. Therefore this code is not uniquely decodeable.