Introduction to Multimedia Compression – Midterm

11/13/2014

Name: ______________________ ID: ______________

Note: 20 points for each problem.

1. Suppose we have a source with a probability model $P = \{p_0, p_1, \ldots, p_m\}$ and entropy H_P. Suppose we have another source with probability model $Q = \{q_0, q_1, \ldots, q_m\}$ and entropy H_Q, where

$$q_i = p_i,\ i = 0, 1, \ldots, j-2, j+1, \ldots, m$$

and

$$q_j = q_{j-1} = (p_j + p_{j-1})/2.$$

How is H_Q related to H_P (greater, equal, or less)? Prove your answer.

[Solution]

Problem 4.

$$\begin{aligned} H_Q - H_P &= -\sum_{i=1}^{m} q_i \log_2 q_i + \sum_{i=1}^{m} p_i \log_2 p_i \\ &= -q_{j-1}\log_2 q_{j-1} - q_j \log_2 q_j + p_{j-1}\log_2 p_{j-1} + p_j \log_2 p_j \end{aligned}$$

Given a function

$$f_a(x) = -x\log x - (a-x)\log(a-x)$$

we can easily show that $f_a(x)$ is maximum for $x = \frac{a}{2}$ Let

$$p_{j-1} + p_j = c$$

then

$$q_{j-1} = q_j = \frac{c}{2}$$

Then

$$\begin{aligned} H_Q - H_P &= -\frac{c}{2}\log_2\frac{c}{2} - \frac{c}{2}\log_2\frac{c}{2} + p_j\log_2 p_j + (c-p_j)\log_2(c-p_j) \quad (1) \\ &= f_c(\frac{c}{2}) - f_c(p_j) \\ &\geq 0 \end{aligned}$$

Therefore $H_Q \geq H_P$. #

2. A source has symbol probabilities $p(a) = 0.4$, $p(b) = 0.1$, $p(c) = 0.3$, and $p(d) = 0.2$.

a) Find a Huffman code for the source.

b) Design a 4-bit Tunstall code for the source.

[Solution]

a)

Symbol	Step 1	Step 2	Step 3	Code
a	0.4	0.4	0.6	1
c	0.3	0.3	0.4	00
d	0.2	0.3		010
b	0.1			011

b)

Initial list:

Letter	Prob.
a	0.4
b	0.1
c	0.3
d	0.2

First iteration:

Letters	Prob.
b	0.1
c	0.3
d	0.2
aa	0.16
ab	0.04
ac	0.12
ad	0.08

Second iteration:

Letter	Prob.
b	0.1
d	0.2
aa	0.16
ab	0.04
ac	0.12
ad	0.08
cb	0.03
cc	0.09
cd	0.06
caa	0.048
cab	0.012
cac	0.036
cad	0.024

Third iteration:

Letter	Prob.	Code
b	0.1	0000
aa	0.16	0001
ab	0.04	0010
ac	0.12	0011
ad	0.08	0100
cb	0.03	0101
cc	0.09	0110
cd	0.06	0111
da	0.08	1000
db	0.02	1001
dc	0.06	1010
dd	0.04	1011
caa	0.048	1100
cab	0.012	1101
cac	0.036	1110
cad	0.024	1111

#

3. Given a number a in the interval [0, 1) with an n-bit binary representation $[b_1 b_2 \ldots b_n]$, show that for any other number b to have a binary representation with $[b_1 b_2 \ldots b_n]$ as the prefix, b has to lie in the interval $[a, a + 1/2^n)$.

[Solution]

The number a can be expressed as:

$$a = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_n \cdot 2^{-n}.$$

If b also has a binary representation $[b_1 b_2 \ldots b_n]$ as prefix, then

$$b = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + \ldots + b_n \cdot 2^{-n} + b_{n+1} \cdot 2^{-(n+1)} + \ldots$$

Therefore,

$$b - a = b_{n+1} \cdot 2^{-(n+1)} + \ldots$$

Obviously $b - a \geq 0$ and $b \geq a$.

On the other hand,

$$\begin{aligned} b - a &= b_{n+1} 2^{-(n+1)} + b_{n+2} 2^{-(n+2)} + \ldots \\ &\leq 2^{-(n+1)} + 2^{-(n+2)} + \ldots \\ &< \tfrac{1}{2^n}. \end{aligned}$$

Therefore, $b < a + 1/2^n$.

4. A sequence is encoded using the LZW algorithm and the initial dictionary shown in the following table.

Index	Entry
1	*a*
2	*c*
3	*r*
4	*y*
5	*Δ*

The output of the LZW encoder is the following sequence: 1, 5, 2, 1, 3, 5, 9, 3, 1, 4, 7. Please decode the message.

[Solution]

The decoded dictionary is:

Index	Entry
1	*a*
2	*c*
3	*r*
4	*y*
5	*Δ*
6	*aΔ*
7	*Δc*
8	*ba*
9	*ar*
10	*rΔ*
11	*Δa*
12	*arr*
13	*ra*
14	*ay*
15	*yΔ*

The decoded message is *aΔcarΔarrayΔc*.

5. We try to encode the sequence *catΔateΔhat* using *ppma* with maximal context length $N = 1$ and an integer arithmetic code with a word length of 6. The alphabet set is $\{h, e, t, a, c, \Delta\}$. Assume that we have finished encoding of *catΔateΔ* and obtained the following context table:

order	context	symbol occurrence counts							Total
		h	*e*	*t*	*a*	*c*	*Δ*	<*ESC*>	
1	*c*	0	0	0	1	0	0	1	2
	a	0	0	2	0	0	0	1	3
	t	0	1	0	0	0	1	1	3
	Δ	0	0	0	1	0	0	1	2
	e	0	0	0	0	0	1	1	2
0	/	0	1	2	2	1	2	1	9
–1	/	1	1	1	1	1	1	0	6

Note that in the context table, the cumulative count is calculated from left-to-right. For example, the zero-order context has cumulative count as follows: $h = 0$, $e = 1$, $t = 3$, $a = 5$, $c = 6$, and $\Delta = 8$.

The transmitted sequence after encoding of *cat*Δ*ate*Δ is 1100001111101100000100 and the current lower and upper bounds are $l = 011100$ and $u = 110011$. Please encode the next letter h and write down the newly transmitted bits for h and the updated lower and upper bounds.

Hint: For an integer AC implementation, the message interval can be updated by:

$$l^{(n)} = l^{(n-1)} + \lfloor (u^{(n-1)} - l^{(n-1)} + 1) \times cum\_count(x_n - 1)/total\_count \rfloor,$$
$$u^{(n)} = l^{(n-1)} + \lfloor (u^{(n-1)} - l^{(n-1)} + 1) \times cum\_count(x_n)/total\_count \rfloor - 1.$$

[Solution]

No h in 1st order context of Δ, encode <*ESC*>, $l = 101000$, and $u = 110011 \rightarrow E_2$ scale.

Transmitted sequence: ****1.

Updated bounds: $l = 010000$, $u = 100111 \rightarrow E_3$ scale.

Updated bounds: $l = 000000$, $u = 101111$, E_3 count = 1.

No 0 order context for h, encode <*ESC*>, $l = 101010$, $u = 101111 \rightarrow E_2$ scale.

Transmitted sequence: ****110.

Updated bounds: $l = 010100$, $u = 011111$, E_3 count = 0 $\rightarrow E_1$ scale.

Transmitted sequence: ****1100.

Updated bounds: $l = 101000$, $u = 111111 \rightarrow E_2$ scale.

Transmitted sequence: ****11001.

Updated bounds: $l = 010000$, $u = 111111$.

Use –1 order context to encode h, $l = 010000$, $u = 010111 \rightarrow E_1 + E_2 + E_1$ scales.

Transmitted sequence: ****11001010.

Updated bounds: $l = 000000$, $u = 111111$. #