Fourier Series Methods†

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12/9/2019

† Chapter 11.1 ~ 11.3 in the textbook.

Orthogonal Functions & Inner Product

- Vectors in linear algebra are not just n-tuples

- If $\mathbf{u}$ and $\mathbf{v}$ are two n-tuple vectors in 3D-space, then the inner product $(\mathbf{u}, \mathbf{v})$ possesses the following properties:
  - $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$ (inner product is commutable)
  - $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$ (k is a scalar)
  - $(\mathbf{u}, \mathbf{u}) = 0$, if $\mathbf{u} = 0$ and $(\mathbf{u}, \mathbf{u}) > 0$, if $\mathbf{u} \neq 0$
  - $(\mathbf{u}+\mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$ (inner product is distributable)

- The definite integral of two functions, f_1 and f_2, over an interval $[a, b]$ possesses the same properties as well $\rightarrow$ we can define "inner product" for functions

Inner Product of Functions

- The inner product of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx$$

- Two vectors are "orthogonal" if the inner product is zero $\rightarrow$ function inner product should be defined similarly: two functions f_1 and f_2 are orthogonal on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0$$

Orthonormal Set of Functions

- A set of real-valued functions $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is said to be orthogonal on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)dx = 0, \quad m \neq n.$$

- The norm of a function ϕ is defined as $\|\phi\| = (\phi, \phi)^{1/2}$. That is,

$$\|\phi(x)\| = \sqrt{\int_a^b \phi^2(x)dx}.$$

If $\{\phi_n\}$ is an orthogonal set on $[a, b]$ and $\|\phi_n\| = 1, \forall n$, then $\{\phi_n\}$ is an orthonormal set of functions on $[a, b]$.

Orthogonal Series Expansion

- If $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on the interval $[a, b]$, is it possible to determine a set of coefficients c_n, $n = 0, 1, 2, \ldots$ such that

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x) + \ldots ?$$

To find the coefficient of ϕ_n, we compute (f, ϕ_n)

$$(f, \phi_n) = c_0(\phi_0,\phi_n) + c_1(\phi_1,\phi_n) + \ldots + c_n(\phi_n,\phi_n) + \ldots$$

Since $\{\phi_n(x)\}$ is an orthogonal set, $(\phi_m,\phi_n) = 0, \forall\ m \neq n$. Therefore,

$$c_n = \frac{(f,\phi_n)}{\|\phi_n\|^2} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \frac{(f,\phi_n)}{\|\phi_n\|^2}\phi_n(x).$$

Completeness of an Orthogonal Set

- In previous discussion, we have $f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n\|^2} \phi_n(x)$, if $f(x)$ can be represented as a linear combination of $\phi_0(x) \sim \phi_\infty(x)$ in the vector space S.

  However, not every functions in S can be represented as a linear combinations of the functions in $\{\phi_n(x)\}$. This is true only when $\{\phi_n(x)\}$ is a complete set of S, i.e., when $\{\phi_n(x)\}$ is a vector basis of S.

Periodic External Forces

- Recall that a linear 2^{nd}-order DE:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = f(t),$$

  where $f(t)$ stands for the external force imposed on the (undamped) system. Often, $f(t)$ is a periodic function (over an interval of interest).

- Question: Is there a systematic way to represent a general periodic function?
  - Well, Taylor series may work, but can we do better?

Properties of a Periodic Function

- **Definition:** The function $f(t)$ defined for all t is said to be periodic provided that there exists a positive number p such that $f(t+p)=f(t)$ for all t. If p is the smallest number with this property, then p is called the period of the function f.

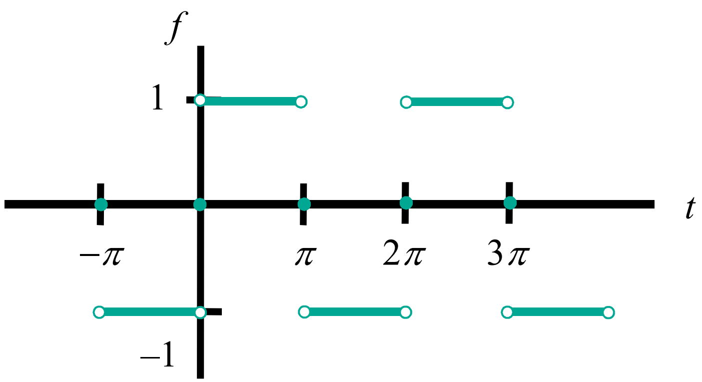
- Remarks:
  - Linear combinations of two (or more) periodic functions will still be a periodic function.
  - If we use a set of periodic functions as basis functions to represent other periodic functions, they should work better than if we use $\{1, x, x^2, x^3, \ldots\}$, as in Taylor series.

Selection of Periodic Basis

- In 1822, J. Fourier asserted that every function $f(t)$ with period 2π can be represented as a linear combination of $\sin nt$ and $\cos nt$, as follows:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right),$$

- Really? How about the function:



Fourier Series

- Note that the set of trigonometric functions:

  $$\{1, \cos t, \cos 2t, \cos 3t, \ldots, \sin t, \sin 2t, \sin 3t, \ldots\}$$

  are orthogonal on the interval $[-\pi, \pi]$.
- The Fourier series of $f(t)$ on $[-\pi, \pi]$ is defined as :

where $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right),$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

The projection vector of $f(t)$ onto $\cos nt$ is $\frac{\langle f(t),\ \cos nt \rangle}{\|\cos nt\|^2}\cos nt$. Thus, $a_n = \frac{\langle f(t),\ \cos nt \rangle}{\|\cos nt\|^2}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt.$$

Fourier Series with Period $2p$

- Note that the set of trigonometric functions

$$\left\{1, \cos\frac{\pi x}{p}, \cos\frac{2\pi x}{p}, \cos\frac{3\pi x}{p}, \cdots, \sin\frac{\pi x}{p}, \sin\frac{2\pi x}{p}, \sin\frac{3\pi x}{p}, \cdots\right\}$$

  is orthogonal on the interval $[-p, p]$.
- The Fourier Series of a function $f(x)$ on $(-p, p)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)$$

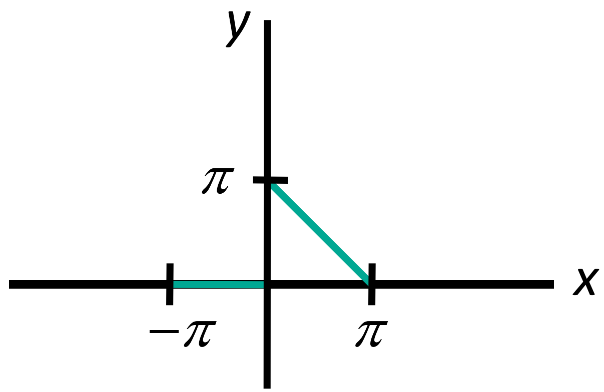
$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx, \quad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx$$

Example: $f(x)=\begin{cases}0, & -\pi<x<0\\ \pi-x, & 0\le x<\pi\end{cases}$ (1/2)

- Since $p=\pi$, we have

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$
$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}(\pi-x)dx\right]$$
$$=\frac{1}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}=\frac{\pi}{2}$$



Example: $f(x)=\begin{cases}0, & -\pi<x<0\\ \pi-x, & 0\le x<\pi\end{cases}$ (2/2)

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\,dx+\int_{0}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[(\pi-x)\frac{\sin nx}{n}\bigg|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\sin nx\,dx\right]$$

$$=-\frac{1}{n\pi}\frac{\cos nx}{n}\bigg|_0^{\pi}=\frac{-\cos n\pi+1}{n^2\pi}$$

(Note that $\cos n\pi=(-1)^n$)

$$=\frac{1-(-1)^n}{n^2\pi}$$

$$b_n=\frac{1}{\pi}\int_0^{\pi}(\pi-x)\sin nx\,dx=\frac{1}{n}$$

$$f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{1}{n}\sin nx\right\}$$

Fourier Convergence Theorem

- **Theorem**: Let f and f' be piecewise continuous on the interval $(-p, p)$; that is f and f' be continuous except at a finite number of points, then the Fourier series of f converges to f at a point of continuity.

  At a point of discontinuity the Fourier series converges to the average:

  $$\frac{f(x+) + f(x-)}{2},$$

  where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and the left, respectively

Example: Converges at Discontinuity

- The following function is discontinuous at $x = 0$:

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

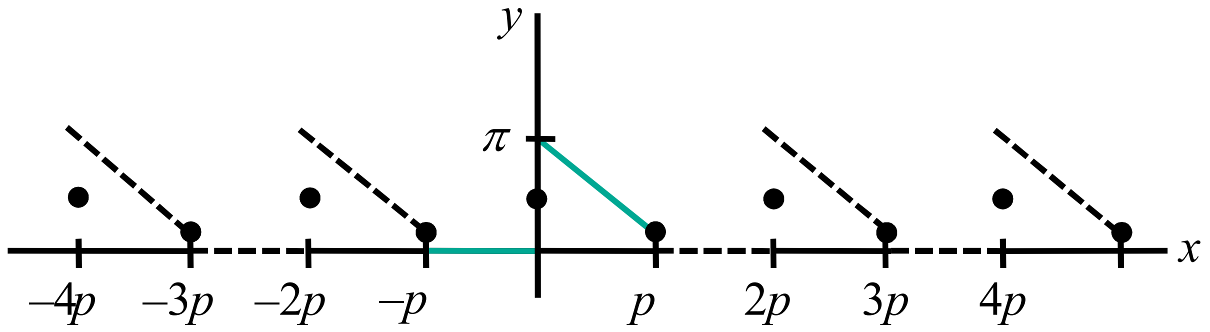
- The series converges to f at $x \neq 0$. At $x = 0$, the series converges to:

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

Periodic Extension

- Fourier series not only represents a function f on the interval $(-p, p)$, but also gives the periodic extension of f outside the interval.

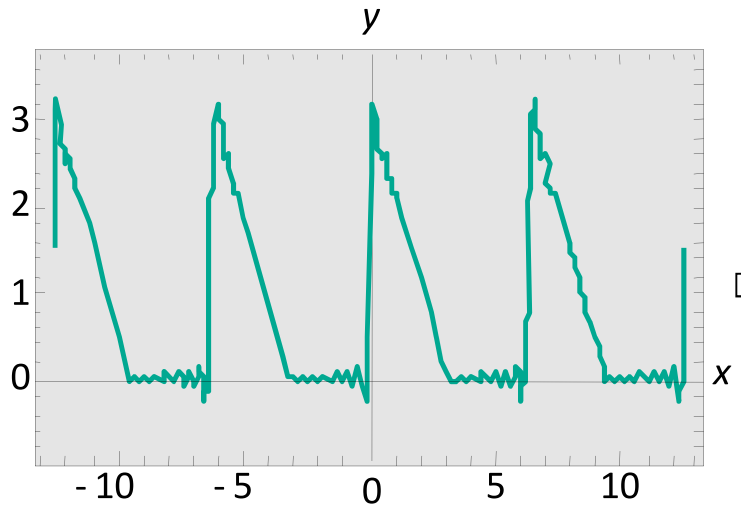
- When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series converges to the average $[f(-p-) + f(-p+)]/2 = [f(p-) + f(-p+)]/2$ at the end points:



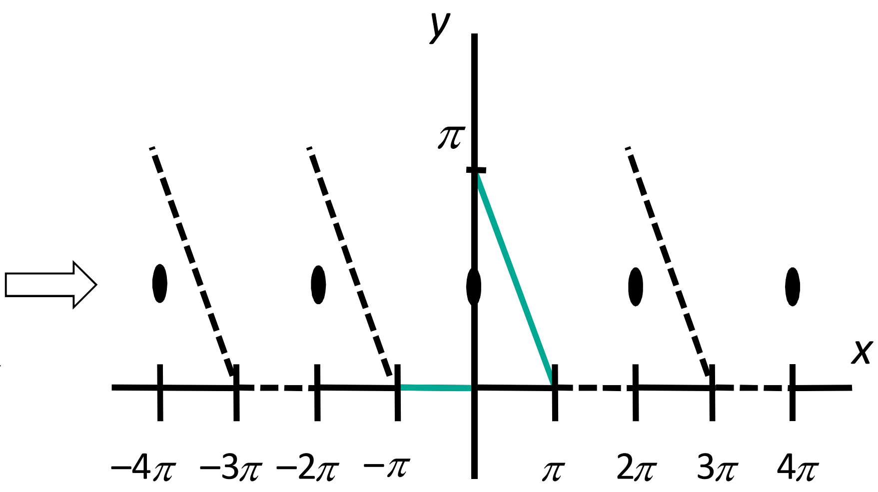
Sequence of Partial Sums (1/2)

- It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example,

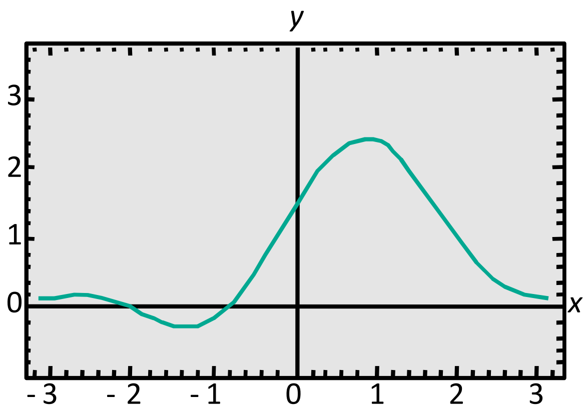
$$S_1(x) = \frac{\pi}{4},\ S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x,\ \cdots$$



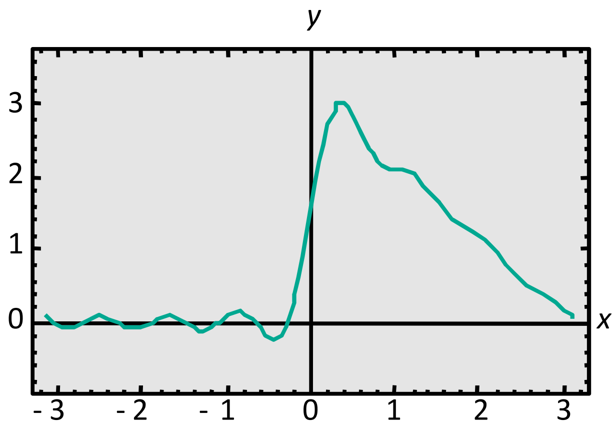
$S_{15}(x)$ on $(-4\pi, 4\pi)$



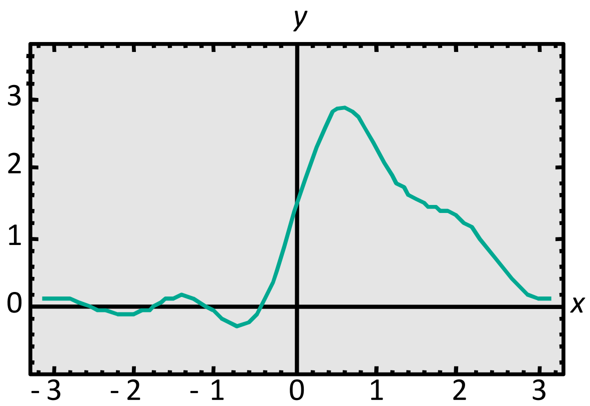
Sequence of Partial Sums (2/2)



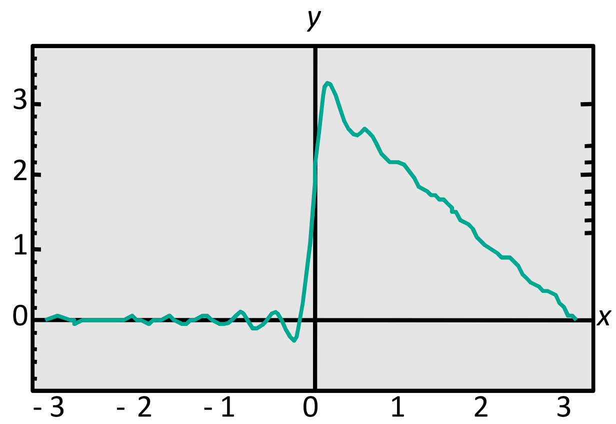
(a) $S_3(x)$ on $(-\pi, \pi)$



(c) $S_8(x)$ on $(-\pi, \pi)$



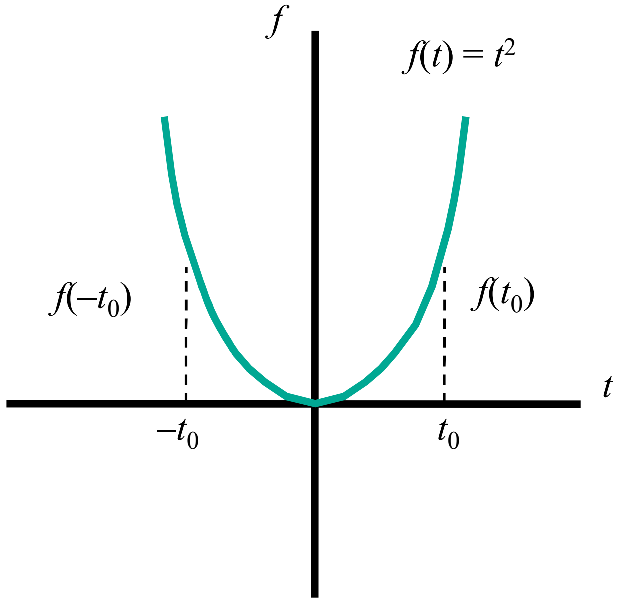
(b) $S_5(x)$ on $(-\pi, \pi)$

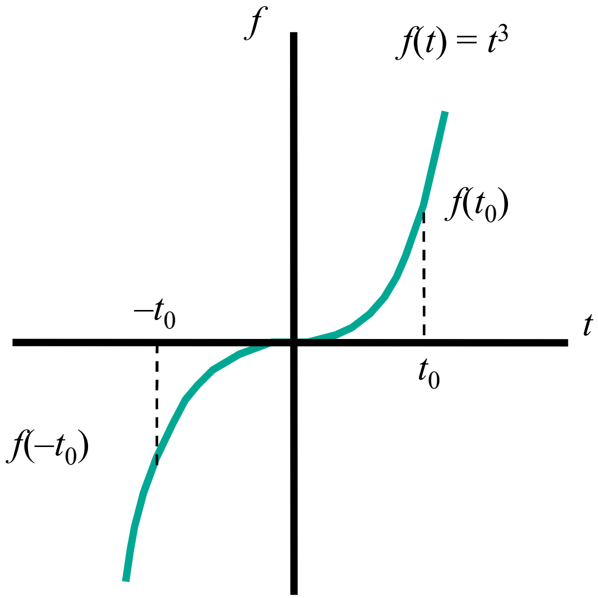


(d) $S_{15}(x)$ on $(-\pi, \pi)$

Even and Odd Functions

- A function is said to be "even" if $f(-t) = f(t)$ and "odd" if $f(-t) = -f(t)$.
- Note that $\cos t$ is even while $\sin t$ is odd.





Properties of Even/Odd Functions

- The product of two even functions is even
- The product of two odd functions is even
- The product of an even and an odd functions is odd
- The sum (difference) of two even functions is even
- The sum (difference) of two odd functions is odd
- If f is even, then

$$\int_{-a}^{a} f(t)dt = 2\int_{0}^{a} f(t)dt.$$

- If f is odd, then

$$\int_{-a}^{a} f(t)dt = 0.$$

Cosine and Sine Series

- If f is an even function on $(-p, p)$, then

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}xdx,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}xdx = 0.$$

- Similarly, if f is odd,

$$a_n = 0, n = 0,1,2,\ldots, \quad b_n = \frac{2}{p}\int_{0}^{p} f(x)\sin\frac{n\pi}{p}xdx.$$

Fourier Cosine and Sine Series

- Suppose that the function $f(x)$ is piecewise continuous on the interval $[0, p]$. The Fourier cosine series of f is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x, \quad \text{with} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx\,.$$

The Fourier sine series of f is:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x, \quad \text{with} \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dt\,.$$

Example: $f(x)=\begin{cases}-1, & -\pi<x<0\\ 0, & x=0,\pi,-\pi\\ +1, & 0<x<\pi\end{cases}$

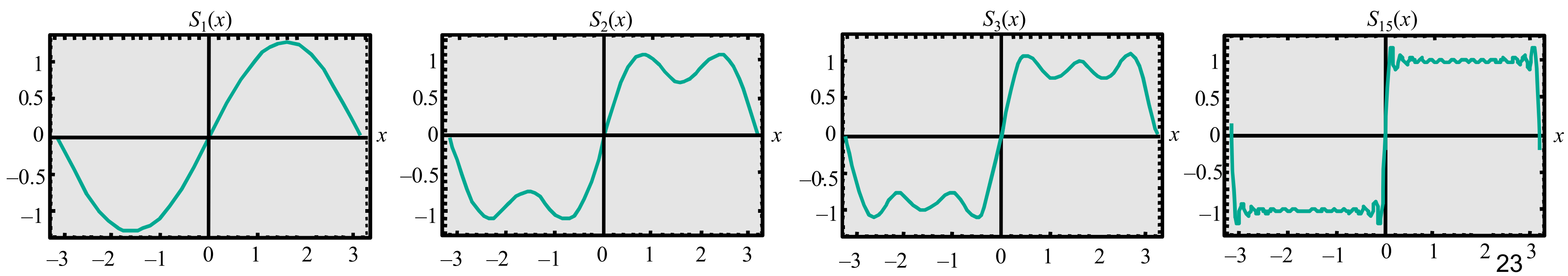
- Calculate b_n as follows:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\cdots$$

$$=\frac{1}{\pi}\left[\frac{1}{n}\cos nx\right]_{-\pi}^{0}+\frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_{0}^{\pi}=\frac{2\left[1-(-1)^n\right]}{n\pi}.$$

$$f(x)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}=\frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right).$$

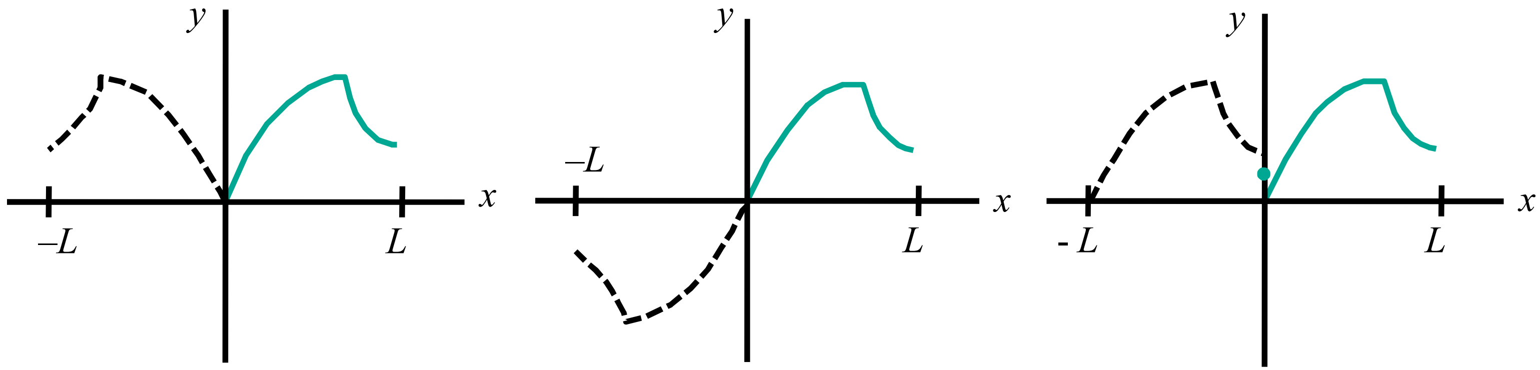
The partial sum tends to overshoot the limiting values of $f(x)$ → Gibbs's Phenomenon:



Half-Range Expansions

- Sometimes, we only care about the Fourier series defined on $(0, L)$. We can define the function f on $(-L, 0)$ so that the expansion has a simpler form.

- Three possible choices of extension:



The third one has period L, others have period $2L$.

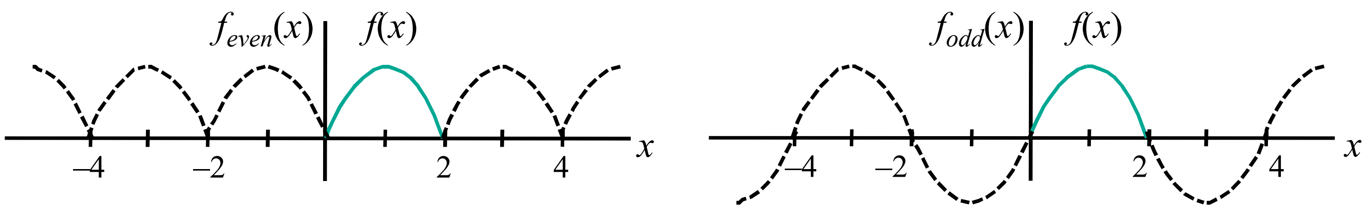
Example: $f(x) = 2x - x^2,\ x \in (0, 2)$

- We can expand $f(x)$ to the range $(-2, 2)$ and make it an even ($f_{\text{even}}(x)$) or an odd ($f_{\text{odd}}(x)$) function:

$$f_{\text{even}}(x) = f(-x) = 2(-x) - (-x)^2 = -2x - x^2, \text{ for } x < 0,$$

or

$$f_{\text{odd}}(x) = -f(-x) = -[2(-x) - (-x)^2] = 2x + x^2, \text{ for } x < 0.$$

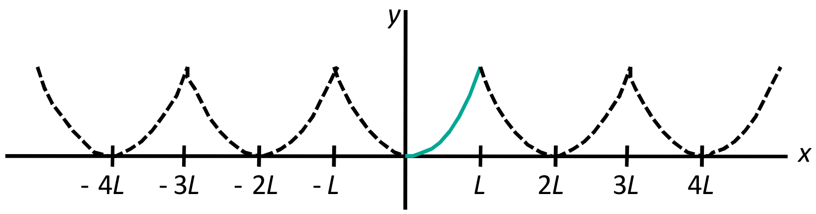


(a) Even expansion of $f(x)$ (b) Odd expansion of $f(x)$

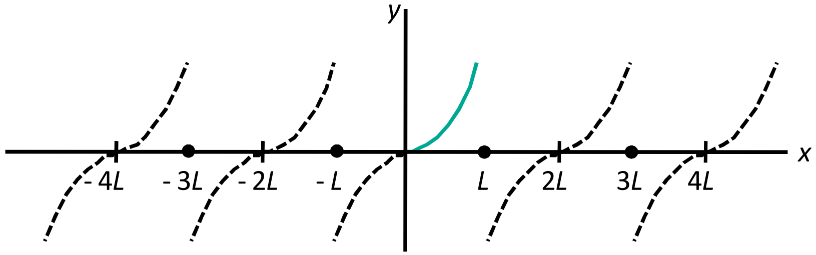
The Fourier expansion of $f_{\text{even}}(x)$ has only cosine terms while $f_{\text{odd}}(x)$ has only sine terms.

Example: $f(x) = x^2,\ 0 < x < L$

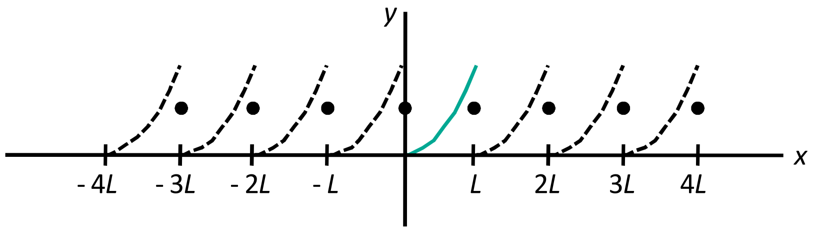
- Expand $f(x)$ in a (a) cosine, (b) sine, (c) Fourier series



(a) Cosine series



(b) Sine series



(c) Fourier series

Review: Periodic Driving Force

- When the driving force $f(t)$ of a DE is periodic and defined over $[0, p]$, Half-range expansion of Fourier series are quite useful. For example, the particular solution of the DE:

$$m\frac{d^2x}{dt^2} + kx = f(t)$$

can be solved by first representing $f(t)$ by a half-range sine expansion and assume a particular solution of the form:

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{p}t$$

Example: $x''+4x = 4t$, $x(0)=x(1)=0$ (1/2)

- Assume that $0 < t < 1$ for $f(t)$, we can use odd extension with $p = 1$ to get the Fourier sine series of f:

$$4t = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi t.$$

The solution $x(t)$ should be in sine series form as well:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t.$$

Note that $x(t)$ satisfies the boundary conditions. Substitute the solution into the DE, we have

$$\sum_{n=1}^{\infty}(-n^2\pi^2 + 4)b_n \sin n\pi t = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi t.$$

Example: $x''+4x = 4t,\ x(0)=x(1)=0$ (2/2)

- The solution of the coefficients b_n is then

$$b_n = \frac{8\cdot(-1)^{n+1}}{n\pi(4-n^2\pi^2)}.$$

The Fourier series solution can be expressed as:

$$x(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi t}{n(4-n^2\pi^2)}, \quad (0 \le t \le 1).$$

which is equivalent to

$$x(t) = t - \frac{\sin 2t}{\sin 2}$$

in the interval $(-1, 1)$.