Power Series Methods†

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† Chapter 6 in the textbook.

Power Series

- A power series in $(x-a)$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

Such a series is said to be a power series centered at a.

- A power series is convergent at a value $x \in I$ if the limit of partial sums exists, i.e.

$$f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$$

The interval of convergence, I, of a power series is the set of all numbers where the series converges.

Ratio Test

- Convergence of a power series can often be checked by the ratio test: suppose $c_n \neq 0$ for all n in

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

and that

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L.$$

If $L < 1$, the series converges absolutely; if $L > 1$ the series diverges; and if $L = 1$ the test is inconclusive.

Radius of Convergence (1/2)

- A power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence ρ, such that $f(x)$ converges for $|x-a| < \rho$ and diverges for $|x-a| > \rho$.
- If $\rho > 0$, $f(x)$ converges for $|x-a| < \rho$, and diverges for $|x-a| > \rho$. If $f(x)$ converges only at its center a, then $\rho = 0$. If it converges for all x, $\rho = \infty$.
- The ratio test is inconclusive at an end point $a \pm \rho$.

Radius of Convergence (2/2)

- Given the power series $g(x) = \sum_{n=0}^{\infty} c_n x^n$, if the limit

$$\rho = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

exists, then

  - If $\rho = 0$, then $g(x)$ diverges for all $x \neq 0$.
  - If $0 < \rho < \infty$, $g(x)$ converges if $|x| < \rho$, and diverges if $|x| > \rho$.
  - If $\rho = \infty$, then $g(x)$ converges for all x.

Power Series of a Function $f(x)$

- The Taylor series of a function $f(x)$ is defined as

$$y = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

  If y converges to $f(x)$ for all x in some open interval containing a, then we say that the function $f(x)$ is analytic at $x = a$.
  - Polynomials are analytic; a rational function is analytic wherever the denominator is not zero.
- Arithmetic of power series
  - The operations of addition, multiplication, and division can be applied to power series as in polynomials.
  - If f and g are analytic at a, so are $f+g$, $f \cdot g$, and f/g (if $g(a) \neq 0$).

Examples of Power Series

- By Taylor ($\forall\ a$) or Maclaurin ($a = 0$) series expansions, common functions can be written in power series forms:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n!} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Example: Adding Two Power Series

- Write $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$ as one power series.

Solution:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2\cdot 1\cdot c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + c_{k-1}]x^k.$$

Power Series Method

- The power series method for solving a DE consists of substituting the power series

$$y = \sum_{n=0}^{\infty} c_n x^n$$

in the DE and determining the coefficients c_0, c_1, … so that the equation satisfies.

- If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then $f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$.

- If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, for all x in the interval of convergence, then $a_n = b_n$ for all $n \geq 0$.

Example: $y' + 2y = 0$

- Since $y = \sum_{n=0}^{\infty} c_n x^n$, and $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$,

  we have

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0.$$

  Perform change of index n to align x^n,

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0 \;\rightarrow\; \sum_{n=0}^{\infty} \left[(n+1)c_{n+1} + 2c_n\right] x^n = 0.$$

  We have a recurrence relation $c_{n+1} = -2c_n/(n+1)$, $n \geq 0$.

$$\rightarrow c_n = (-2)^n c_0/n!,\ n \geq 1 \;\rightarrow\; y(x) = \sum_{n=0}^{\infty} \frac{(-2)^n c_0}{n!} x^n.$$

Power Series Solutions

- Suppose the linear 2nd-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

is put into the standard form

$$y'' + P(x)y' + Q(x)y = 0,$$

then:

A point $x = x_0$ is said to be an ordinary point of the DE if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is said to be a singular point of the equation.

Example: Ordinary, Singular Points

- Every finite value of x is an ordinary point of

$$y'' + (e^x)\, y' + (\sin x)\, y = 0.$$

- $x = 0$ is a singular point of the DE

$$y'' + (e^x)\, y' + (\ln x)\, y = 0.$$

Polynomial-Coefficient DEs

- Recall that a polynomial is analytic at any value x, and a rational function is analytic except at points where its denominator is zero. Thus, a 2nd-order polynomial-coefficient DE has singular points when $a_2(x) = 0$, since

$$P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)}.$$

- Examples
  - The Euler equation $ax^2y'' + bxy' + cy = 0$ has a singular point at $x = 0$.
  - The equation $(x^2 + 1)y'' + xy' - y = 0$ has singular points at $x = \pm i$.

Solutions Near an Ordinary Point

- **Theorem:** If $x = x_0$ is an ordinary point of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

we can always find two linearly independent solutions in the form of a power series centered at x_0, that is,

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

A series solution converges at least on some interval defined by $|x - x_0| < \rho$, where ρ is the distance from x_0 to the nearest singular point.

- Note that for $|x - x_0| \geq \rho$, $y(x)$ may or may not converge. Further investigations are required.

Ex: $(x^2 - 4)y'' + 3xy' + y = 0,\ y(0) = 4,\ y'(0) = 1$

- Note that the singular points are ± 2, there should be a solution with radius of convergence at least 2.

  Since

  $$y = \sum_{n=0}^{\infty} c_n x^n,\quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1},\ \text{and}\ \ y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

  Therefore,

  $$\sum_{n=2}^{\infty} n(n-1)c_n x^n - 4\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

  add dummy terms for $n = 0, 1$ — change of index n — add dummy term $n = 0$

  $$\sum_{n=0}^{\infty} n(n-1)c_n x^n - 4\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + 3\sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

  Combining the terms, we have $c_{n+2} = \dfrac{(n+1)c_n}{4(n+2)},\ n \geq 0.$

Ex: $(x^2 - 4)y'' + 3xy' + y = 0$ (2/2)

- When $n = 0, 2, 4, \ldots$, we have

$$c_2 = \frac{c_0}{4\cdot 2},\ c_4 = \frac{3c_0}{4^2\cdot 2\cdot 4},\ c_6 = \frac{3\cdot 5c_0}{4^3\cdot 2\cdot 4\cdot 6}, \cdots \to c_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{4^n\cdot 2\cdot 4\cdots(2n)}c_0.$$

When $n = 1, 3, 5, \ldots$, we have

$$c_3 = \frac{2c_1}{4\cdot 3},\ c_5 = \frac{2\cdot 4c_1}{4^2\cdot 3\cdot 5},\ c_7 = \frac{2\cdot 4\cdot 6c_1}{4^3\cdot 3\cdot 5\cdot 7}, \cdots \to c_{2n+1} = \frac{2\cdot 4\cdot 6\cdots(2n)}{4^n\cdot 1\cdot 3\cdots(2n+1)}c_1.$$

Therefore, the solution is

$$y(x) = c_0\left(1 + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{3n}n!}x^{2n}\right) + c_1\left(x + \sum_{n=1}^{\infty}\frac{n!}{2^n\cdot 1\cdot 3\cdots(2n+1)}x^{2n+1}\right).$$

and $y(0) = c_0 = 4$, $y'(0) = c_1 = 1$.

Translated Series Solutions

- If the initial condition is given at x_0 other than zero, we have to assume the general solution form

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

  This way, we can obtain the IVP solution with $y(x_0) = c_0$ and $y'(x_0) = c_1$ easily.

- As an alternative, we can translate the equation by letting $t = x - x_0$ and assume the solution form:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

Ex: $(t^2-2t-3)y''+3(t-1)y'+y = 0,\ y(1) = 4,\ y'(1) = -1$

- Perform a change of variable to the DE by $x = t - 1$:

$$t^2 - 2t - 3 = (x + 1)^2 - 2(x + 1) - 3 = x^2 - 4,$$

and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx},\quad \frac{d^2y}{dt^2} = \left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]\frac{dx}{dt} = \frac{d^2y}{dx^2}.$$

Hence, the DE becomes $(x^2-4)d^2y/dx^2 + 3x(dy/dx) + y = 0$.
→ Same DE as the previous one.

Substituting $x = t-1$ into the previous solution, we get

$$y(t) = 4 + (t-1) + \frac{1}{2}(t-1)^2 + \frac{1}{6}(t-1)^3 + \frac{3}{32}(t-1)^4 + \cdots$$

which converges if $-1 < t < 3$.

Example: $y'' + (\cos x)\, y = 0$ (1/2)

- Since

$$y'' + (\cos x)y$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\sum_{n=0}^{\infty} c_n x^n$$

$$= 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(c_0 + c_1 x + c_2 x^2 + \cdots\right)$$

$$= 2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \cdots = 0$$

Example: $y'' + (\cos x)\, y = 0$ (2/2)

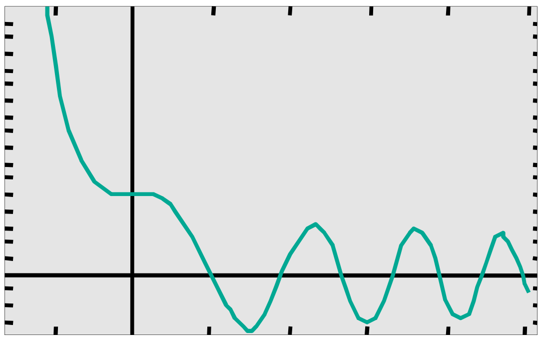
We have:

$$2c_2 + c_0 = 0\,,\;\; 6c_3 + c_1 = 0\,,\;\; 12c_4 + c_2 - \frac{1}{2}c_0 = 0\,,\;\; \cdots$$

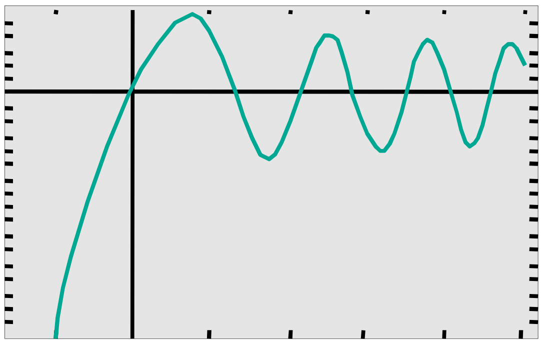
Therefore:

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots,\;\; y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots$$

with region of convergence $|x| < \infty$.



(a) Plot of $y_1(x)$ vs. x



(b) Plot of $y_2(x)$ vs. x

Solutions about Singular Points

- Let $y(x) = \Sigma c_n x^n$, if $P(x)$ is not analytic at 0, its power series form $\Sigma b_k x^k$ will not converge to $P(0)$ at 0 given any b_k. However, it is possible that

  $$(\Sigma b_k x^k)\,(\Sigma n c_n x^{n-1})$$

  may still converge to $P(x)y'(x)$.

  In short, even if $x = 0$ is a singular point, the power series expression of

  $$y'' + P(x)y' + Q(x)y$$

  may still converge to zero.

Regular Singular Points

- Assume that a DE in the standard form

  $y'' + P(x)y' + Q(x)y = 0$ has a singular point at x_0.

  If there are two functions $p(x)$ and $q(x)$, both are analytic at x_0, such that $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$, the original DE can be rewritten as:

$$y'' + \frac{p(x)}{(x - x_0)}y' + \frac{q(x)}{(x - x_0)^2}y = 0,$$

  then, we call $x = x_0$ a regular singular point of the DE.

  Otherwise, $x = x_0$ is a irregular singular point.

Remarks on Singularity of P and Q

- If $x = 0$ is a singular point, the power series expansion of $P(x)$ at 0 approaches ∞.
- However, if $P(x)$ grows slower than $1/x$ when $x \to 0$, then $xP(x)$ is convergent. That is, $p(x) = xP(x)$ is analytic at 0. Similarly, $q(x)$ is analytic at 0 if $Q(x)$ grows slower than $1/x^2$.
- Note that, for the DE $y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0,$

  $x = 0$ is a regular singular point if $p(x)$ and $q(x)$ are polynomials.

Example: Singular Points

- For $(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0$, $x = 2$ and $x = -2$ are singular points. We have

$$P(x) = \frac{3}{(x-2)(x+2)^2} \quad \text{and} \quad Q(x) = \frac{5}{(x-2)^2(x+2)^2}.$$

- Obviously $x = 2$ is a regular singular point, and $x = -2$ is an irregular singular point.

Example: Non-polynomial $p(x)$, $q(x)$

- The DE $x^4y'' + (x^2 \sin x)y' + (1 - \cos x)y = 0$ can be expressed as

$$y'' + \frac{\sin x / x}{x} y' + \frac{(1-\cos x)/x^2}{x^2} y = 0.$$

Since $x = 0$ is not an ordinary point and

$$p(x) = \frac{\sin x}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots,$$

$$q(x) = \frac{1-\cos x}{x^2} = \frac{1}{x^2}\left[1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) \right] = \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} \cdots,$$

are both analytic (convergent) at 0, thus $x = 0$ is a regular singular point.

Solution near Singular Points

- For a constant-coefficient Cauchy-Euler equation

$$x^2y'' + p_0xy' + q_0y = 0,$$

where p_0 and q_0 are constants, we can assume that $y(x) = x^r$ is a solution $\rightarrow$ r is a root of the equation:

$$r(r-1) + p_0r + q_0 = 0.$$

If we have coefficient functions $p(x)$ and $q(x)$ instead, is it possible that

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

is a solution?

Method of Frobenius

- If $x = x_0$ is a regular singular point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r},$$

where the number r is a constant to be determined. The series will converge on some interval of $0 < x - x_0 < R$.

Example: $3xy'' + y' - y = 0$ (1/3)

- Solution: let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, we have

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \text{ and } y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}.$$

Therefore,

$$3xy'' + y' - y$$

$$= x^r \left[r(3r-2)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(3k+3r+1)c_{k+1} - c_k]x^k \right] = 0$$

We have: $\begin{cases} r(3r-2)c_0 = 0 \\ (k+r+1)(3k+3r+1)c_{k+1} - c_k = 0, \ k = 0, 1, 2, \cdots \end{cases}$

Example: $3xy'' + y' - y = 0$ (2/3)

- Hence,

$$\rightarrow \begin{cases} r = 0, 2/3 \\ c_{k+1} = \dfrac{c_k}{(k+r+1)(3k+3r+1)}, \ k = 0, 1, 2, \cdots \end{cases}$$

Substituting $r = 0$ and $r = 2/3$ into the recurrence eq.,

$$\rightarrow \begin{cases} r = 2/3, \ c_{k+1} = \dfrac{c_k}{\big(3(k+1)+2\big)(k+1)} \rightarrow c_n = \dfrac{c_0}{n!\, 5\cdot 8\cdot 11\cdots(3n+2)} \\ r = 0, \quad c_{k+1} = \dfrac{c_k}{(k+1)\big(3(k+1)-2\big)} \rightarrow c_n = \dfrac{c_0}{n!\, 1\cdot 4\cdot 7\cdots(3n-2)} \end{cases}$$

Example: $3xy'' + y' - y = 0$ (3/3)

- Let $c_0 = 1$, we have two series solutions

$$\begin{cases} y_1(x) = x^{2/3}\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5\cdot 8\cdot 11\cdots(3n+2)} x^n\right] \\ y_2(x) = x^{0}\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,1\cdot 4\cdot 7\cdots(3n-2)} x^n\right] \end{cases}.$$

Since $y_1(x)$ and $y_2(x)$ are linearly independent on the entire axis, $y(x) = k_1 y_1(x) + k_2 y_2(x)$ is the general solution of the DE on any interval not containing the origin (note that 0^0 is undefined).

Indicial Equation

- The equation derived from the coefficient of the smallest degree of x in the Frobenius method is the indicial equation.

- The solutions of the indicial equation with respect to r are called the indicial roots.

Frobenius Series Solutions (1/2)

- **Theorem:** If $x = 0$ is a regular singular point of

$$x^2y'' + xp(x)y' + q(x)y = 0.$$

Let $\rho > 0$ denote the minimum of the radii of convergence of the power series

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n .$$

Let r_1 and r_2 be the (real) roots, with $r_1 \geq r_2$, of the indicial equation

$$r(r - 1) + p_0 r + q_0 = 0.$$

Then, we have the following properties:

Frobenius Series Solutions (2/2)

1. For $x > 0$, there exists a solution of the form
$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad (a_0 \neq 0)$$
corresponding to the larger root r_1.
2. If $r_1 \neq r_2$ and $r_1 - r_2 \notin Z^+$, then there exists a 2nd linearly independent solution for $x > 0$ of the form
$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n \quad (b_0 \neq 0)$$
corresponding to the smaller root r_2.
3. The radii of convergence of the solutions are at least ρ, the nearest distance to the nearby singular point.

Example: $2xy'' + (1 + x)\, y' + y = 0$

- Since $x^2y'' + \tfrac{1}{2} \cdot x(1+x)\, y' + \tfrac{1}{2}\, xy = 0$, $p(x) = (1+x)/2$ and $q(x) = x/2 \rightarrow p_0 = \tfrac{1}{2}$ and $q_0 = 0 \rightarrow r^2 - r/2 = 0, \rightarrow r = 0, \tfrac{1}{2}$.

  For $r_1 = \tfrac{1}{2}$, let $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$, then $a_n = \frac{(-1)^n a_0}{2^n n!}$.

  For $r_2 = 0$, let $y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$, we have

$$b_n = \frac{(-1)^n b_0}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}.$$

  The general solution is $y = c_1 y_1(x) + c_2 y_2(x)$.

Example: $xy'' + 2y' + xy = 0$ (1/3)

- When $r_1 - r_2$ is a positive integer, the Frobenius solution is only guaranteed for r_1. However, in this example, we still have two solutions even if $r_1 - r_2 = 1$.

  The DE can be written as $y'' + \frac{2}{x} y' + \frac{x^2}{x^2} y = 0.$

  The indicial equation $r(r-1) + 2r = 0$ has roots $0, -1$. Start with $r_2 = -1$, we have

  $$y(x) = x^{-1} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n-1}.$$

  Hence,

  $$\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$ → Check the coefficients of x^{-2} and x^{-1}!

Example: $xy'' + 2y' + xy = 0$ (2/3)

The first two terms gives us $0 \cdot c_0 = 0$ and $0 \cdot c_1 = 0$, which means c_0 and c_1 can be arbitrary constants.
Thus, the recurrence relation $c_n = -c_{n-2}/n(n-1)$, $n \geq 2$ can be divided into two groups of coefficients:

$$c_{2n} = \frac{(-1)^n c_0}{(2n)!} \quad \text{and} \quad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}, \quad \text{for} \quad n \geq 1.$$

Therefore, a general solution is

$$y(x) = x^{-1} \sum_{n=0}^{\infty} c_n x^n$$

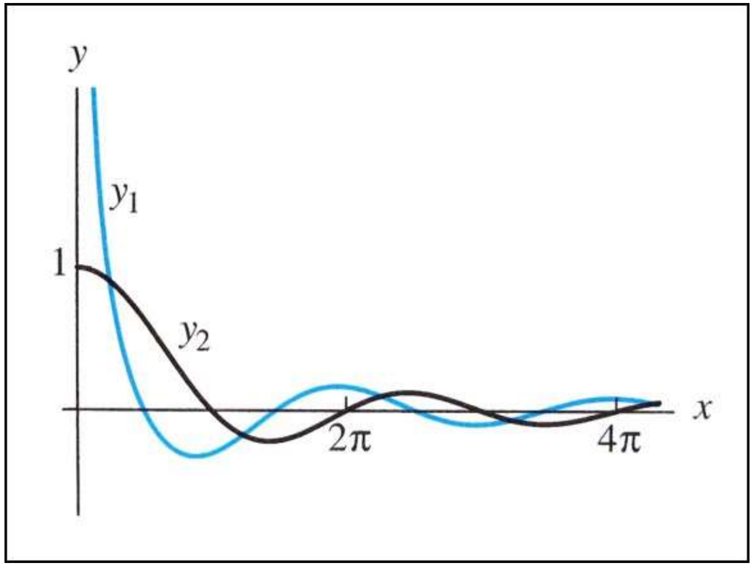
$$= \frac{c_0}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \frac{c_1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example: $xy'' + 2y' + xy = 0$ (3/3)

- Now, if you pay attention, you will recognize that the solution is simply

$$y(x) = x^{-1}(c_0 \cos x + c_1 \sin x).$$

The graph of the solution is:



2nd Solution by Reduction of Order

- If there is only one solution in Frobenius form for

$$y'' + P(x)y' + Q(x)y = 0,$$

we can find the 2nd solution by reduction of order.

Recall that the reduction of order formula tells us

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx.$$

Summary of Indicial Roots (1/2)

- Case I: r_1 and r_2 are distinct, $r_1 - r_2 \neq N$, for some integer $N \rightarrow$ exists two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ and } y_2(x) = \sum_{n=0}^{\infty} c_n x^{n+r_2}.$$

- Case II: $r_1 - r_2 = N$, for some integer $N \rightarrow$ exist two linearly independent solutions of the form

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \\ y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0 \end{cases}.$$

Note that C could be zero.

Summary of Indicial Roots (2/2)

- Case III: If $r_1 = r_2$, there exists two linearly independent solutions of the form

$$\begin{cases} y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0 \\ y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1} \end{cases}.$$

Bessel's Equations

- Bessel's equation of order $v \geq 0$ is defined as

$$x^2y'' + xy' + (x^2 - v^2)y = 0.$$

  The solutions are called Bessel functions of order v.

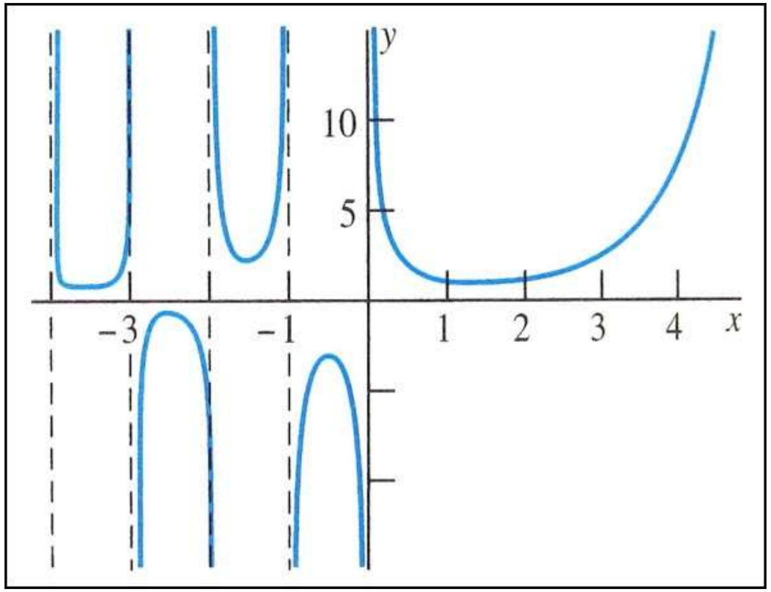
- Bessel's functions first appear in 1764 when Euler was studying the vibration of drum membrane. Later, the functions appears in many physics problems, from fluid equations to planet motions.

Gamma Function $\Gamma(x)$

- The gamma function (or generalized factorial function) is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

  For $x > 0$, we have $\Gamma(x+1) = x\,\Gamma(x)$.



Solution of Bessel's Equation (1/2)

- Let the solution be $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, we have

$$x^2 y'' + xy' + (x^2 - v^2)y =$$

$$c_0(r^2 - v^2)x^r + x^r \sum_{n=1}^{\infty} c_n\left[(n+r)^2 - v^2\right]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}.$$

The indicial equation is $r^2 - v^2 = 0$, pick $r = v$

$$x^v \sum\nolimits_{n=1}^{\infty} c_n n(n+2v)x^n + x^v \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= x^v \left[(1+2v)c_1 x + \sum_{k=0}^{\infty} \left[(k+2)(k+2+2v)c_{k+2} + c_k\right]x^{k+2}\right] = 0.$$

Solution of Bessel's Equation (2/2)

Therefore, we have

$$\begin{cases} (1+2v)c_1 = 0 \\ c_{k+2} = \dfrac{-c_k}{(k+2)(k+2+2v)}, \quad k = 0, 1, 2,\ldots \end{cases}$$

$$\rightarrow \begin{cases} c_1 = c_3 = c_5 = c_7 = \cdots = 0 \\ c_{2n} = \dfrac{(-1)^n c_0}{2^{2n} n!(1+v)(2+v)\cdots(n+v)}, \quad n = 1, 2, 3,\ldots \\ \quad = \dfrac{(-1)^n c_0 \Gamma(v+1)}{2^{2n} n! \Gamma(n+v+1)} = \dfrac{(-1)^n}{2^{2n+v} n! \Gamma(n+v+1)}, \text{ if } c_0 = \dfrac{1}{2^v \Gamma(v+1)} \end{cases}$$

Bessel Functions of the 1st Kind (1/2)

- The solutions of Bessel's Equation can be written as

$$J_v(x) = \sum_{n=0}^{\infty} c_{2n} x^{2n+v} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$

Similarly, starting from $r = -v$, we have

$$J_{-v}(x) = \sum_{n=0}^{\infty} c'_{2n} x^{2n-v} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-v+n)} \left(\frac{x}{2}\right)^{2n-v}.$$

$J_v(x)$ and $J_{-v}(x)$ are called Bessel's functions of the first kind of order v and $-v$.

Bessel Functions of the 1st Kind (2/2)

- Now, we want to find the general solution of the Bessel's DE. Notice that $r_1 - r_2 = 2v$:

1. If $2v \neq$ integer, then $J_v(x)$ and $J_{-v}(x)$ are linearly independent.

2. If $2v = 2m + 1$, m is an integer, then $J_{m+1/2}(x)$ and $J_{-m-1/2}(x)$ are still linearly independent.

3. If $2v = 2m$, m is an integer, then $J_m(x)$ and $J_{-m}(x)$ are linear dependent solutions of Bessel's DE.
   $\rightarrow$ must find another solution!

J_m & J_{-m} are Linearly Dependent (1/2)

- **Proof:**

  Assume that $v = m$ is an integer, we want to show that $J_{-m}(x) = (-1)^m J_m(x)$.

  1) Perform change of index on $J_{-m}(x)$:

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-m+n)} \left(\frac{x}{2}\right)^{2n-m}$$

  Let $2k+m = 2n-m \rightarrow k = n-m$ and $n = k+m$, we have

$$J_{-m}(x) = \sum_{k=-m}^{\infty} \frac{(-1)^{k+m}}{(k+m)!\Gamma(1+k)} \left(\frac{x}{2}\right)^{2k+m}$$

J_m & J_{-m} are Linearly Dependent (2/2)

2) Since $|\Gamma(x)| = \infty$, for $x = 0, -1, -2, \ldots$, we have

$$J_{-m}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+m}}{(k+m)!\Gamma(1+k)} \left(\frac{x}{2}\right)^{2k+m}$$

3) Finally, note that

$$\begin{aligned}(k+m)!\Gamma(1+k) &= [(k+m)(k+m-1) \ldots (k+2)(k+1)]\, k!\, \Gamma(1+k) \\ &= k!\, [(k+m)(k+m-1) \ldots (k+2)]\, \Gamma(2+k) \\ &= k!\, \Gamma(1+m+k).\end{aligned}$$

Therefore,

$$J_{-m}(x) = (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(1+m+k)} \left(\frac{x}{2}\right)^{2k+m} = (-1)^m J_m(x)$$

#

Bessel Functions of the 2nd Kind

- If v is any non-integer number, we can apply linear combinations of $J_v(x)$ and $J_{-v}(x)$ to obtain another solution:

$$y_2(x) \overset{def}{=} Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}.$$

  For $m \in$ integer, $Y_m(x) = \lim_{v\to m} Y_v(x)$ still converges.

- For any non-integer value of v, the general solution of Bessel's DE can also be written as

$$y = c_1 J_v(x) + c_2 Y_v(x).$$

  $Y_v(x)$ is called the Bessel function of the 2nd kind.

Example: The Aging Spring

- The DE for the free undamped motion of a mass on an aging spring is given by: $mx'' + ke^{-\alpha t}x = 0$. The change of variable,

$$s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}\, e^{-\alpha t/2},$$

turns the DE into

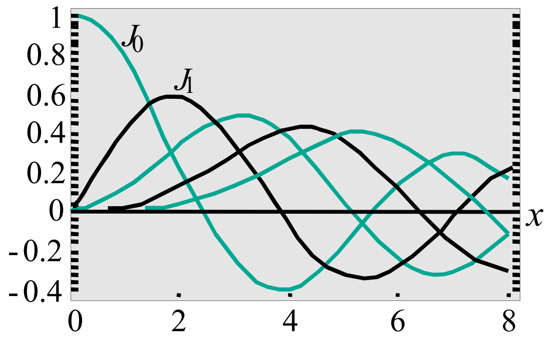
$$s^2 \frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2 x = 0.$$

Therefore, it's the Bessel DE with $v = 0$. The general solution is

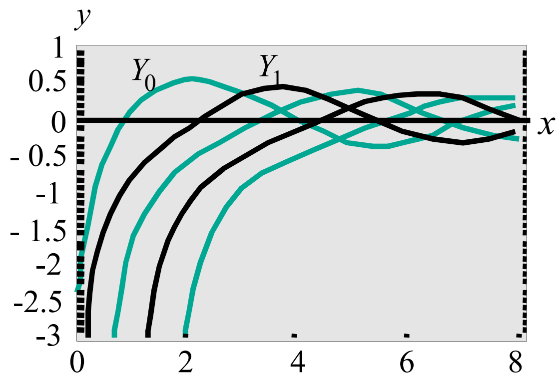
$$x(t) = c_1 J_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right) + c_2 Y_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right).$$

Properties of Bessel Functions

- For $m = 0, 1, 2, \ldots$, we have:
  - $J_{-m}(x) = (-1)^m J_m(x)$
  - $J_m(-x) = (-1)^m J_m(x)$
  - $J_m(0) = 0$ if $m > 0$; $J_m(0) = 1$, if $m = 0$
  - $\lim_{x \to 0^+} Y_m(x) = -\infty$



Bessel functions of the first kind, $n = 0, 1, 2, 3, 4$



Bessel functions of the 2[nd] kind, $n = 0, 1, 2, 3, 4$

Bessel Functions with $v = 0$

- When $v = 0$, we have $J_v(x) = J_{-v}(x)$, the 2nd solution can be obtained by Case III of the method of Frobenius: $y_1(x) = J_v(x)$, and

$$y_2(x) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2} \right)^{2n} \right) \ln(x) + \sum_{n=1}^{\infty} b_n x^n .$$

Substitute $y_2(x)$ into the DE and solve for b_n, we have:

$$y_2(x) = \frac{2}{\pi} J_0(x) \left[\gamma + \ln \frac{x}{2} \right] - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) \left(\frac{x}{2} \right)^{2n}$$

$\gamma = 0.57721566$ is Euler's constant.

Differential Recurrence Relation

- Bessel functions satisfy differential recurrence relations as follows:
  - $xJ_v'(x) = vJ_v(x) - xJ_{v+1}(x)$
  - $xJ_v'(x) = xJ_{v-1}(x) - vJ_v(x)$
- To prove the relations, first, we have to show that

$$\frac{d}{dx}\left[x^v J_v(x)\right] = x^v J_{v-1}(x) \text{ and } \frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x).$$

The recurrence relations can be derived easily, e.g.,

$$\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -vx^{-v-1} J_v(x) + x^{-v} J_v'(x) = -x^{-v} J_{v+1}(x)$$

$$\rightarrow xJ_v'(x) = vJ_v(x) - xJ_{v+1}(x).$$

Differentiation of $x^{v}J_{v}(x)$

- Since

$$J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k+v},$$

then

$$\frac{d}{dx}\left[x^v J_v(x)\right] = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2v}}{2^{2k+v} k!(v+k)\Gamma(v+k)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2v-1}}{2^{2k+v-1} k!\Gamma(v+k)}$$

$$= x^v \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma((v-1)+k+1)} \left(\frac{x}{2}\right)^{2k+(v-1)}$$

$$= x^v J_{v-1}(x)$$

Legendre's Equation

- Legendre's equation of order α is the 2nd-order linear DE of the form

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

where the real number $\alpha > -1$. The only singular points of the Legendre's equation are at $+1$ and -1.

Solution of Legendre's Equation (1/2)

- Since $x = 0$ is an ordinary point of the equation, substitute $y = \Sigma c_m x^m$ into the Legendre's equation, we have

$$c_{m+2} = -\frac{(\alpha - m)(\alpha + m + 1)}{(m+2)(m+1)} c_m \ , \ \ m \geq 0.$$

It can be shown that,

$$c_{2m} = (-1)^m \frac{\alpha(\alpha-2)(\alpha-4)\cdots(\alpha-2m+2)(\alpha+1)(\alpha+3)\cdots(\alpha+2m-1)}{(2m)!} c_0,$$

and

$$c_{2m+1} = (-1)^m \frac{(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)(\alpha+2)(\alpha+4)\cdots(\alpha+2m)}{(2m+1)!} c_1.$$

Solution of Legendre's Equation (2/2)

- If $\alpha = n$, a non-negative integer, we have

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

and

$$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]$$

Notice that if n is an even integer, $y_1(x)$ terminates. When n is an odd integer, $y_2(x)$ terminates.

Legendre Polynomials (1/2)

- The solution polynomial of Legendre equation of order n, with special selection of c_0 (n even) or c_1 (n odd), are called Legendre polynomial of degree n:

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} ,$$

where $N = \lfloor n/2 \rfloor$. For example:

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

Legendre Polynomials (2/2)

They are the solutions of

$$n=0: \quad (1-x^2)y'' - 2xy' = 0$$

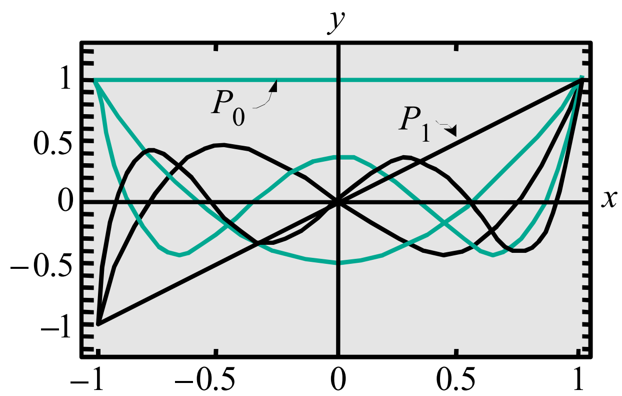
$$n=1: \quad (1-x^2)y'' - 2xy' + 2y = 0$$

$$n=2: \quad (1-x^2)y'' - 2xy' + 6y = 0$$

$$n=3: \quad (1-x^2)y'' - 2xy' + 12y = 0$$

$$\vdots \qquad\qquad \vdots$$

Legendre polynomials are orthogonal over [–1, 1].



Legendre Polynomials, for n = 0, 1, 2, 3, 4