# Modeling with High-Order Differential Equations

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# Linear Models: IVP

- Many linear dynamic systems can be represented using a 2nd order DE with constant coefficients:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = g(t)$$

In this formulation, $g(t)$ is the input or forcing function of the system, the output of the system is a solution $y(t)$ of the DE that satisfies the initial conditions $y(t_0) = y_0,\ y'(t_0) = y_1$ on an interval containing $t_0$.

# Free Undamped Motion

- Hooke's law describes the restoring force:

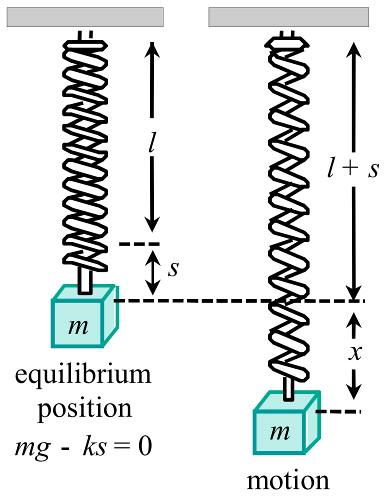
$$F = ks$$

- Newton's 2nd law ($F = ma$) describes the motion:

$$m(d^2x/dt^2) = -k(s + x) + mg$$
$$= -kx + (mg - ks) = -kx$$

- DE of free undamped motion:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \;\; x(0) = x_0, \;\; x'(0) = x_1$$



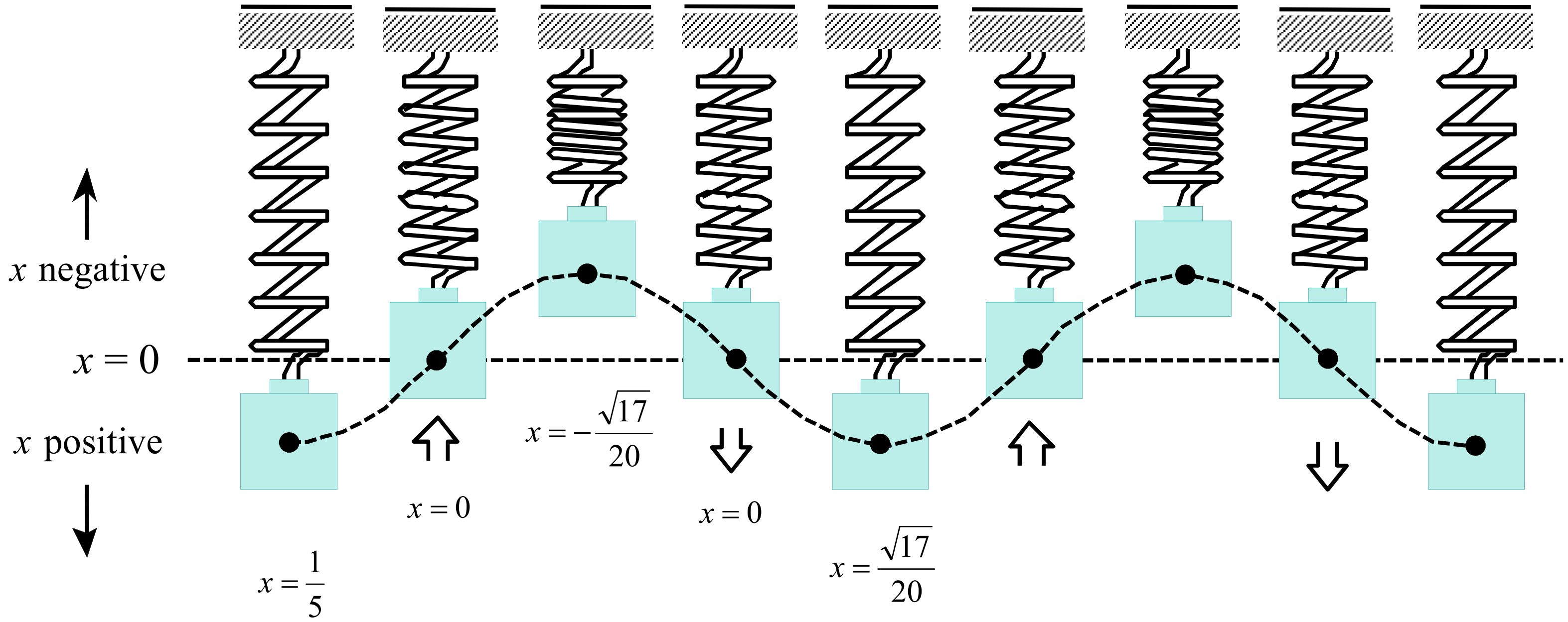

- Solution of the motion:

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

# Alternative Form of Solution

- By applying trigonometric formula, we have:

$$x(t) = A\sin(\omega t + \phi),\ \ A = \sqrt{c_1^2 + c_2^2},\ \ \tan\phi = \frac{c_1}{c_2}$$



# Aging Spring

- In real world, the spring constant $k$ usually varies as the spring gets old. Replace $k$ with $k(t) = ke^{-\alpha t}$, $k > 0$, $\alpha > 0$, we have a more realistic system model:

$$mx'' + ke^{-\alpha t}x = 0$$

$\rightarrow$ Non-constant coefficient 2nd-order linear DE!

# Free Damped Motion

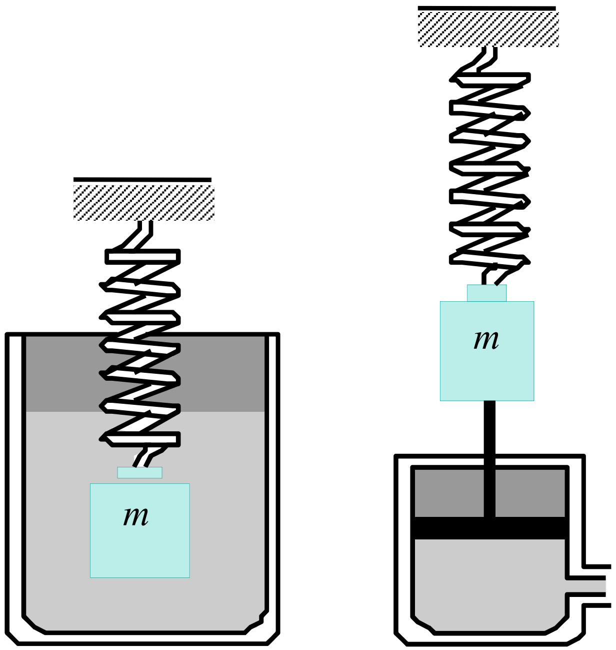
- DE of free damped motion:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

$$\rightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

$\rightarrow$ The roots of the auxiliary eq.:

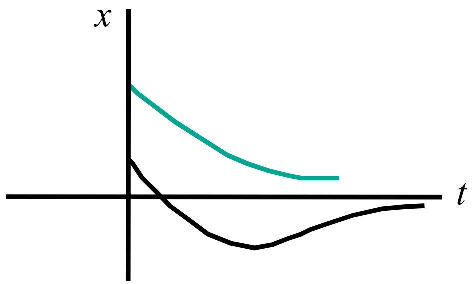
$$m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$



# Three Cases of Damped Motion

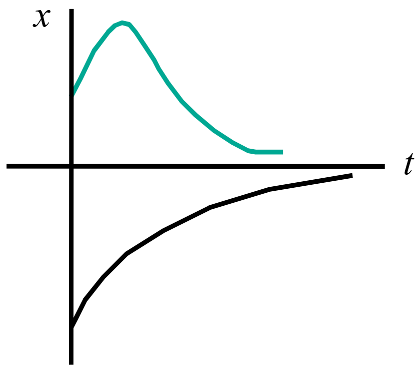
- Case I: Over-damped

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}\,t})$$



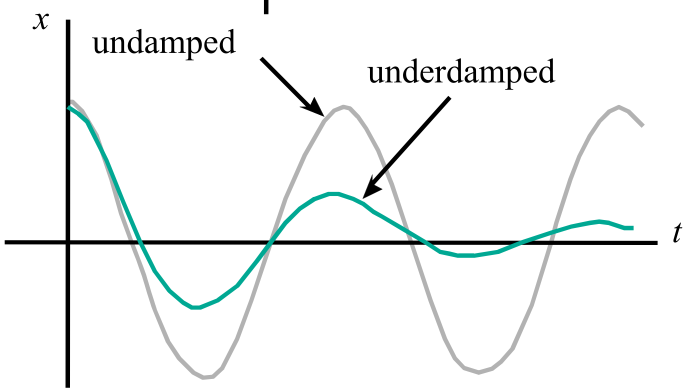

- Case II: Critically damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$




- Case III: Under-damped

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}\; t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\; t)$$

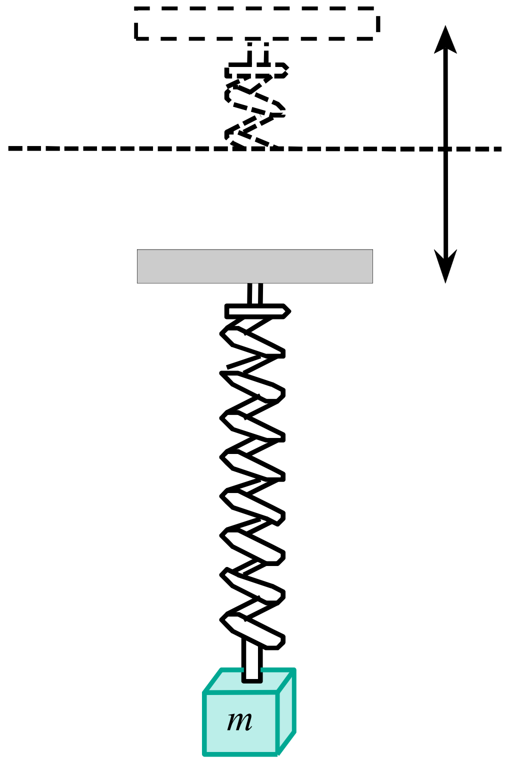


# Driven Motion

- Now, consider the effect of external force $f(t)$ on the damped motion system:

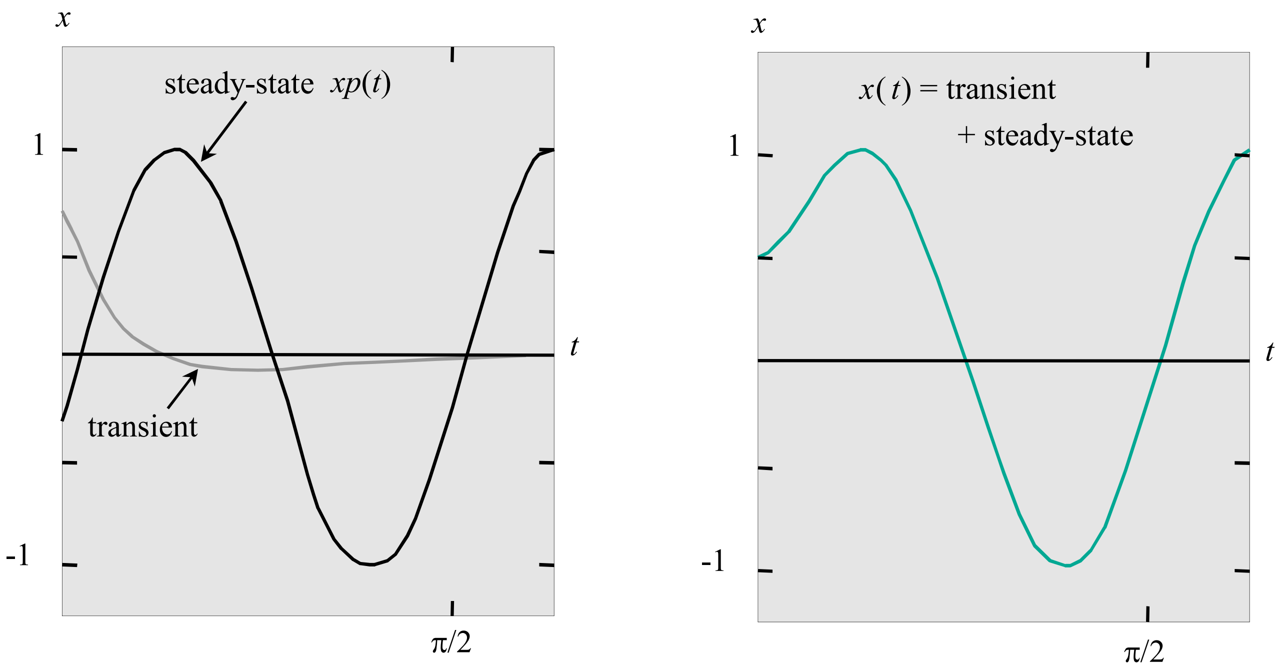
$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$

$$\rightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$



# Transient and Steady-State Terms

- When $F(t)$ is a periodic function and $\lambda > 0$, the solution is the sum of a non-periodic function $x_c(t)$ and a periodic function $x_p(t)$. Moreover $\lim_{t\to\infty} x_c(t) = 0$.

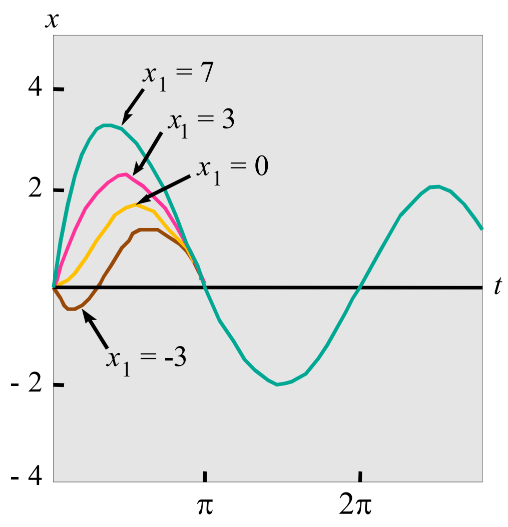


# Example: Transient/Steady State

- The solution of

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 4\cos t + 2\sin t\,, \quad x(0) = 0\,, \quad x'(0) = x_1$$

is $x(t) = \underbrace{(x_1 - 2)e^{-t}\sin t}_{\text{transient}} + \underbrace{2\sin t}_{\text{steady-state}},$



# Undamped Forced Motion

- The solution of

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \gamma t\,, \quad x(0) = 0\,, \quad x'(0) = 0$$

is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \sin \gamma t$$

where $c_1 = 0$, $c_2 = -\gamma F_0 / \omega(\omega^2 - \gamma^2)$.

$$\rightarrow \quad x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)}(-\gamma \sin \omega t + \omega \sin \gamma t)\,, \quad \gamma \neq \omega$$

$\rightarrow$ There is no transient term.

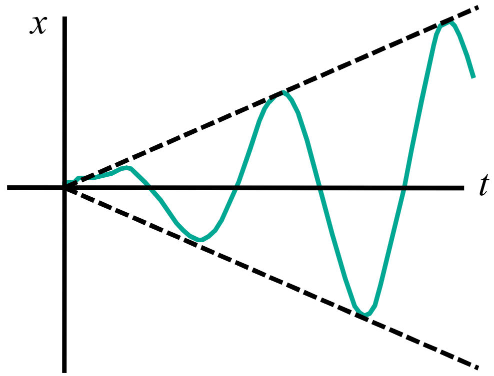
# Pure Resonance

- In the previous example, when $\gamma \to \omega$, the displacement of the system become large as $t \to \infty$.

$$x(t) = \lim_{\gamma \to \omega} F_0 \frac{-\gamma \sin \omega t + \omega \sin \gamma t}{\omega(\omega^2 - \gamma^2)}$$

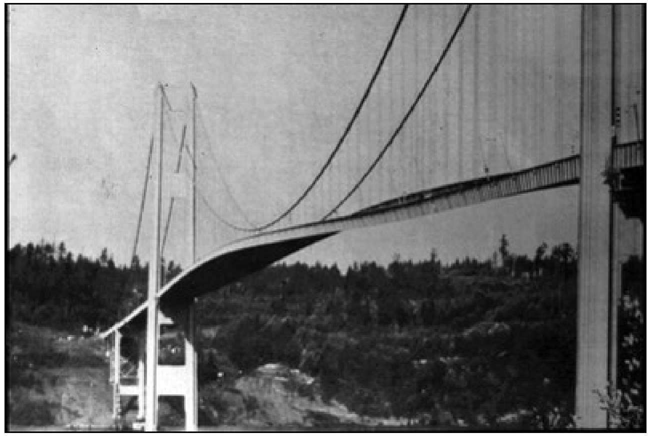
$$= F_0 \lim_{\gamma \to \omega} \frac{\frac{d}{d\gamma}(-\gamma \sin \omega t + \omega \sin \gamma t)}{\frac{d}{d\gamma}(\omega^3 - \omega\gamma^2)}$$

$$= \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t$$



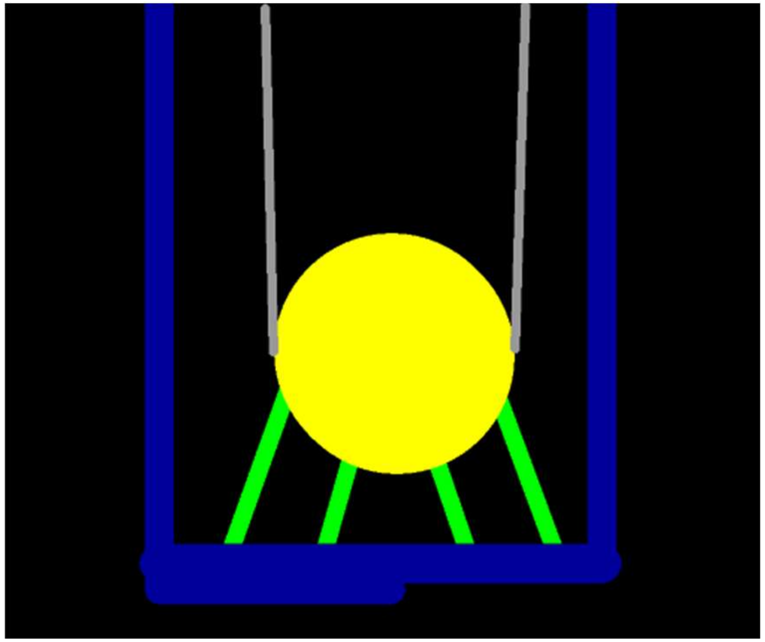
# Tacoma Narrow Bridge, WA, USA

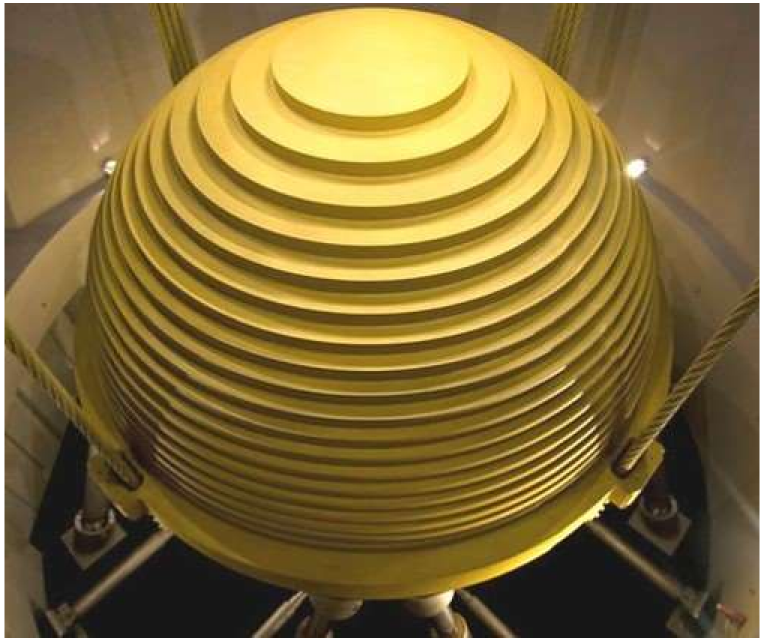
- Opened in July 1, 1940, collapsed in Nov. 7, 1940.
  - The wind-blow frequency matched the natural frequency of the bridge, which caused a pure resonance effect that destroyed the bridge.



# Damping System of Taipei 101

- Taipei 101 uses a 730-ton damping ball† to stabilize the building under wind-blow effect



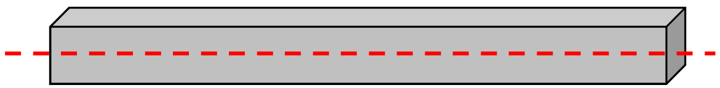


† Picture from: https://nl.m.wikipedia.org/wiki/Bestand:Tuned_mass_damper.gif

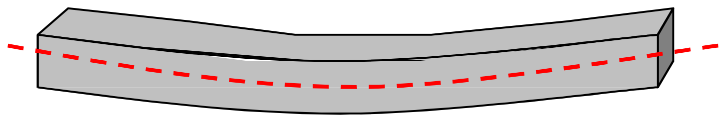
# Linear Models: BVP

- The deflection of a flexible beam can be modelled by a 4th-order differential equation:

$$\underbrace{EI}_{\text{flexural rigidity}}\frac{d^4 y}{dx^4} = \underbrace{w(x)}_{\text{load per unit length}}$$



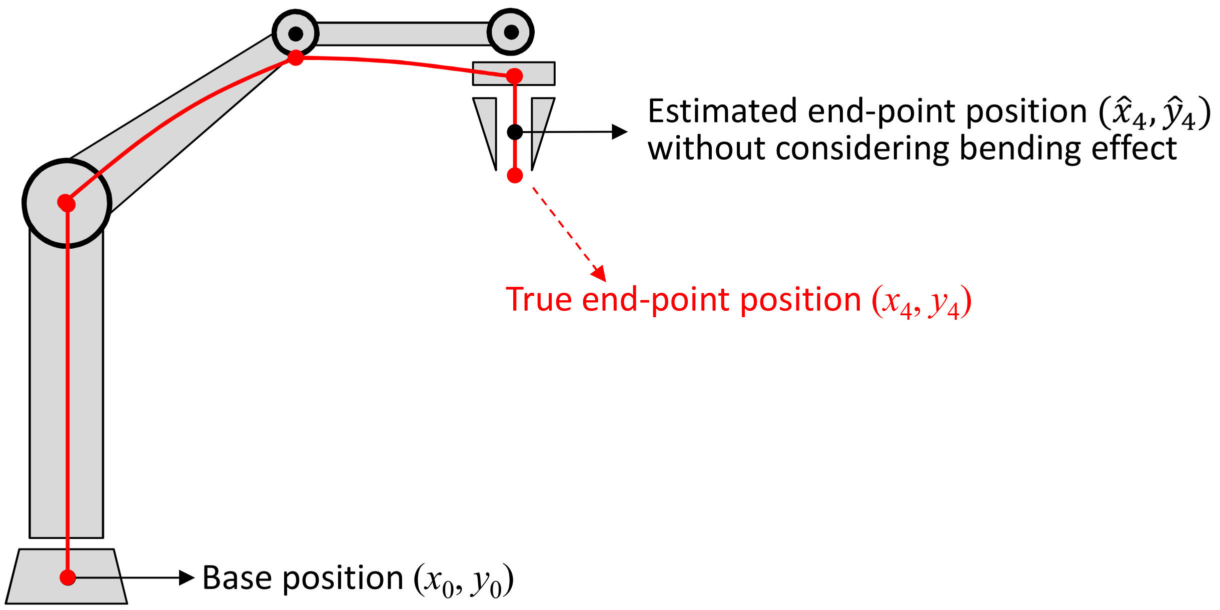
A straight flexible beam



The deflection curve of the beam

# Flexible Beam Applications

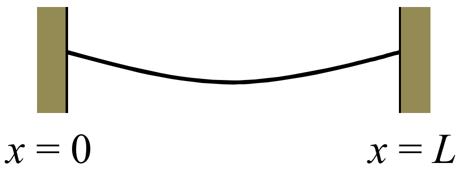
- For precise robot arm control, we must take into account the bending effect of the robot links:



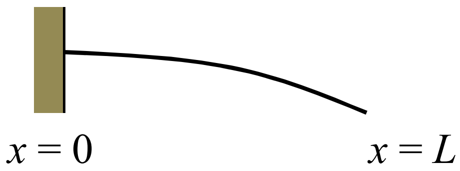
# Boundary Conditions

- Boundary conditions of a flexible beam:

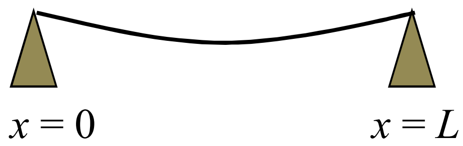
| End of beam | Boundary conditions | |
|---|---|---|
| embedded | $y = 0$ | $y' = 0$ |
| free | $y'' = 0$ | $y''' = 0$ |
| supported | $y = 0$ | $y'' = 0$ |



Embedded at both ends



Free at the right end



Supported at both ends

# Eigenvalue Problems

- An eigenvalue problem in DE is a homogeneous BVP such that the boundary conditions evaluate to $0$ and there is a parameter $\lambda$ at the coefficient of $y$:

  $$y'' + p(x)y' + \lambda q(x)y = 0,\ y(a) = 0,\ y(b) = 0.$$

  The eigenvalue problem tries to find a $\lambda$ (eigenvalue) such that the BVP has a nontrivial solution.

- The non-trivial solution that corresponding to an eigenvalue $\lambda$ is then called an eigenfunction.

# Example: $y'' + \lambda y = 0,\ y(0) = y(L) = 0$ (1/2)

- The problem can be solved by enumerating different cases when $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

(1) $\lambda = 0$, we have $y'' = 0$,
  $\rightarrow$ the general solution is $y(x) = Ax + B$.
  $\rightarrow$ $y = 0$ is the only solution for the BVP
  $\rightarrow$ $\lambda = 0$ is not an eigenvalue of the BVP

(2) $\lambda < 0$, let $\lambda = -\alpha^2$, $\alpha > 0$, we have $y'' - \alpha^2 y = 0$,
  $\rightarrow$ the general solution is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$.
  $\rightarrow$ $y = 0$ is the only solution for the BVP
  $\rightarrow$ $\lambda < 0$ do not have eigenvalues of the BVP

# Example: $y'' + \lambda y = 0,\ y(0) = y(L) = 0$ (2/2)

(3) $\lambda > 0$, let $\lambda = \alpha^2$, $\alpha > 0$, we have $y'' + \alpha^2 y = 0$,

$\rightarrow$ the solution is $y(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$.

$\rightarrow$ $y(0) = 0$ implies $c_1 = 0$

$\rightarrow$ $y(L) = 0$ implies $\sin(\alpha L) = 0$, or $\alpha L = n\pi,\ n \in Z$

$\rightarrow$ The BVP has infinitely many eigenvalues:

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n \in Z$$

and the corresponding eigenfunctions are:

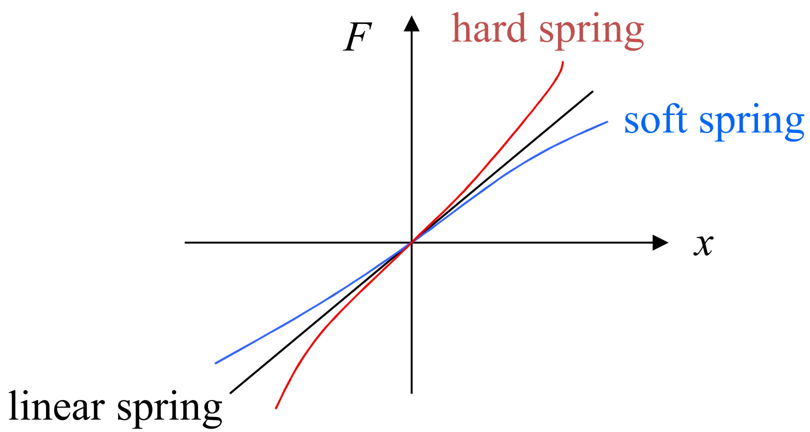
$$y_n = c_2 \sin\left(\frac{n\pi}{L}x\right), \qquad n = 1,\ 2,\ 3,\ \ldots$$

# Nonlinear Spring Models (1/2)

- The general mathematical model of an undamped spring has the form:

$$m\frac{d^2x}{dt^2} + F(x) = 0$$

for a linear spring model, $F(x) = kx$. However, spring are quite often nonlinear, e.g. $F(x) = kx + k_1x^3$.



# Nonlinear Spring Models (2/2)

- Damping force of a spring system can be nonlinear as well:

$$m\frac{d^2x}{dt^2} + \beta\left|\frac{dx}{dt}\right|\frac{dx}{dt} + F(x) = 0$$

- Restoring force $F(x)$ is usually an odd function such as $kx + k_1x^3$. The reason is that we want $F(-x) = -F(x)$.

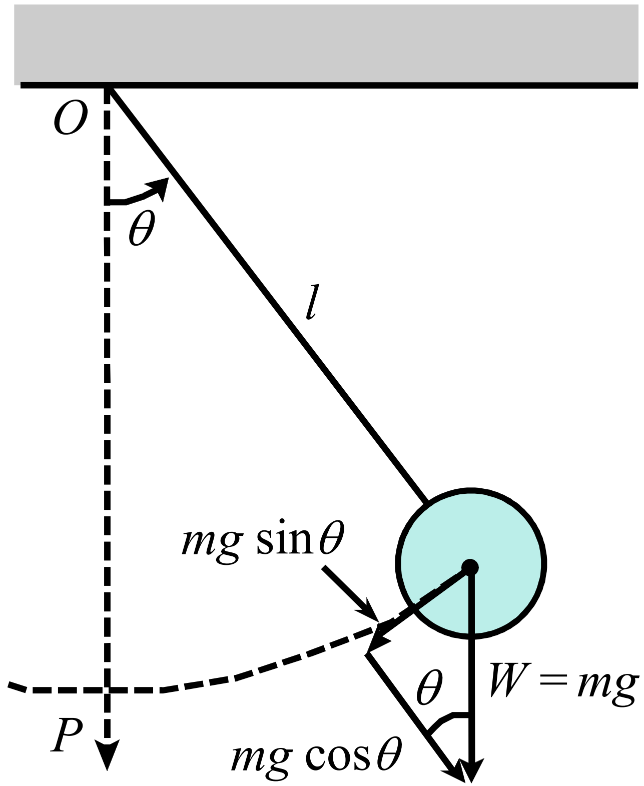
# Nonlinear Pendulum

- The pendulum system can be modeled as

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

  Using Maclaurin series of $\sin\theta$, we have

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$$



# Linearization of Nonlinear Systems

- Assuming that $\sin\theta \approx \theta - \theta^3/6$, we have:

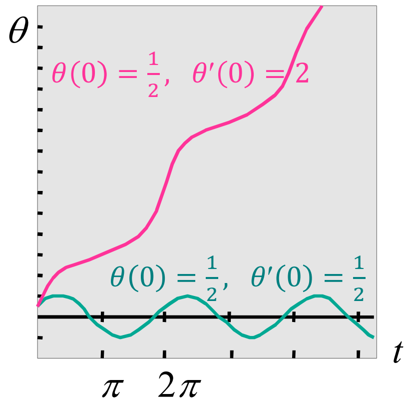
$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l}\right)\theta + \left(\frac{g}{6l}\right)\theta^3 = 0. \quad \rightarrow$$

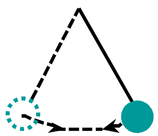
*A nonlinear model similar to the spring systems!*

System can be linearized by assuming $\sin\theta \approx \theta$:

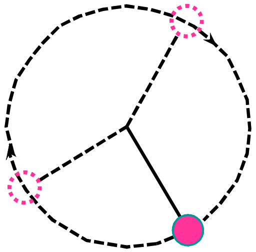
$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{l}\right)\theta = 0.$$

- Impact of initial values:



$\theta(0) = \frac{1}{2}$, $\theta'(0) = \frac{1}{2}$



$\theta(0) = \frac{1}{2}$, $\theta'(0) = 2$