High Order Differential Equations

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Initial-Value Problems

- For a linear n^{th}-order differential equation, an initial-value problem (IVP) is:

  Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  Subject to:

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

Existence of a Unique Solution

- **Theorem**: Let $a_n(x)$, $a_{n-1}(x)$, …, $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the IVP exists on the interval and is unique.

Examples:

- The trivial solution $y = 0$ is the unique solution of the IVP $3y^{(3)} + 5y'' - y' + 7y = 0$, $y(1) = y'(1) = y''(1) = 0$ on any interval containing $x = 1$.

- The solution $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution of the IVP $y'' - 4y = 12x$, $y(0) = 4$, $y'(0) = 1$ on any interval containing $x = 0$.

- The solution family $y = cx^2 + x + 3$ are solutions of the IVP $x^2y'' - 2xy' + 2y = 6$, $y(0) = 3$, $y'(0) = 1$
  $\rightarrow a_2(x) = x^2 = 0$ at 0.

Boundary-Value Problem

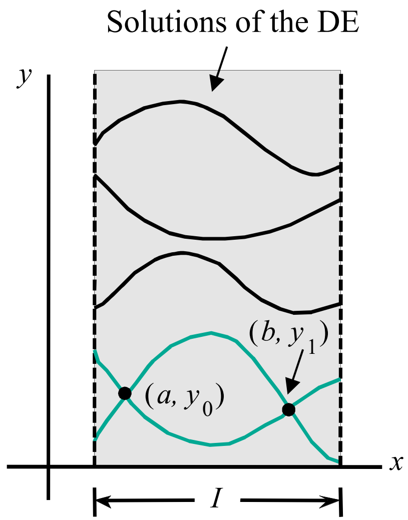
- Solving a linear DE with y or its derivatives specified at different points. For example,

Solve

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Subject to

$$y(a)=y_0,\ \ y(b)=y_1$$

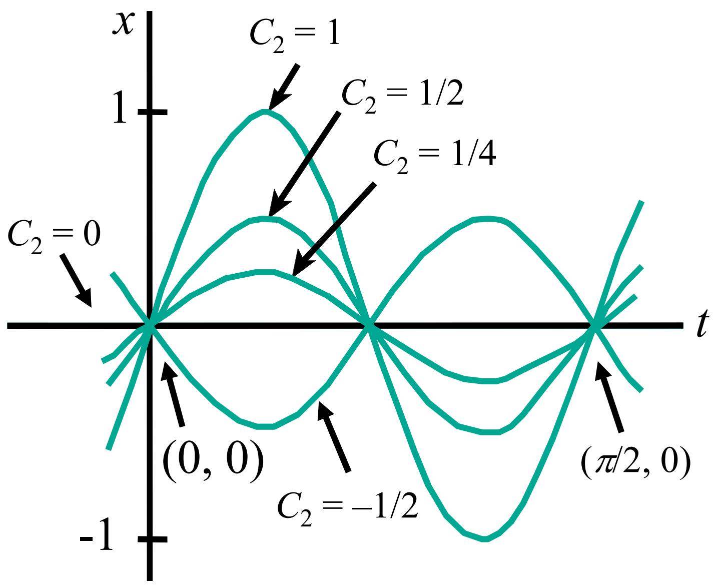


Solutions of a BVP

- A BVP can have many, one, or no solutions.

  Example: $x = c_1 \cos 4t + c_2 \sin 4t$ is a solution family of $x'' + 16x = 0$. What are the solutions of the BVPs with

  (1) $x(0) = 0,\ x(\pi/2) = 0$?
  (2) $x(0) = 0,\ x(\pi/8) = 0$?
  (3) $x(0) = 0,\ x(\pi/2) = 1$?



Homogeneous Equations

- For a linear n^{th}-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

  if $g(x) = 0$, it is called a homogeneous differential equation, otherwise, it is non-homogeneous.

  Note that: the solution of a non-homogeneous differential equation is based on the solution to its associated homogeneous differential equation.

Differential Operators

- The symbol D, defined by $Dy = dy/dx$, is called a **differential operator**. D transforms a function into another function.

  Example: $D(\cos 4x) = -4\sin 4x$, $D(5x^3 - 6x^2) = 15x^2 - 12x$

- Polynomial expressions involving D, such as $D + 3$ and $D^2 + 3D - 4$ are also differential operators.

Linear Operator

- An n^{th}-order differential operator is defined as:

$$L = a_n(x)D^n + a_{n\text{-}1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x)$$

L is a linear operator, that is,

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta L(g(x))$$

"D" Representation of DEs

- Any differential equations can be expressed in terms of the D notation.

  For example, $y'' + 5y' + 6y = 5x - 3$ can be written as $D^2y + 5Dy + 6y = 5x - 3$ or $(D^2 + 5D + 6)y = 5x - 3$

- A linear n^{th}-order differential equation can be write compactly as $L(y) = g(x)$.

Superposition Principle

- **Theorem**: Let $y_1, y_2, \ldots, y_k$ be solutions of a homogeneous linear n^{th}-order DE on an interval I. Then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_k y_k(x),$$

where c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on this interval.

$\rightarrow$ Can be proved by using linear operator property.

Linear Dependency

- A set of functions $f_1(x),\ f_2(x),\ \ldots,\ f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0.$$

  Otherwise, it's said to be linearly independent.

- Example: Are $\cos^2 x$, $\sin^2 x$, $\sec^2 x$, $\tan^2 x$ linearly dependent on the interval $(-\pi/2, \pi/2)$?

Wronskian

- We are interested in linearly independent solutions of a linear differential equations $\rightarrow$ How to verify?

- Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ has at least $n\text{–}1$ derivatives. The determinant

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_n^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of the functions.

Criterion for Linear Independency

- **Theorem**: Let $y_1, y_2, \ldots, y_n$ be n *solutions* of the linear nth-order homogeneous DE on an interval I. Then, the set of solutions is linearly independent on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

- Example: for $y'' - 3y' + 2y = 0$, the two solutions $y_1 = e^x$ and $y_2 = e^{2x}$ has the Wronskian:

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0, \ \forall x \in (-\infty, \infty).$$

Wronskian Independence Checks

- The previous theorem implies that if y_1 and y_2 are two solutions of a linear homogeneous D.E., then either $W(y_1, y_2) \equiv 0$ or $W(y_1, y_2) \neq 0, \forall x$.
  - This can be proven by applying the existence and uniqueness theorem on zero initial condition IVP!

- For any two functions y_1 and y_2 that are not solutions of a linear homogeneous D.E. over an interval I:
  - If $W(y_1, y_2) \neq 0$, for some $x \in I$, then y_1 and y_2 are linearly independent over I.
  - If $W(y_1, y_2) \equiv 0$, $\forall x$, and y_1 & y_2 are nonzero with continuous derivatives in I, then y_1 and y_2 are linearly dependent over I.

Fundamental Set of Solutions

- Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear n^{th}-order DE on an interval I is said to be a fundamental set of solutions on the interval I.

- **Theorem**: There exists a fundamental set of solutions for the homogeneous linear n^{th}-order DE.

  $\rightarrow$ Similar to that a vector can be decomposed into linear combinations of basis vectors.

General Solution (1/2)

- **Theorem**: Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear n^{th}-order DE on an interval I. Then, the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x),$$

where c_i, $i = 1,2,\ldots,n$ are arbitrary constants.

Proof on $n = 2$:
Let Y be a solution of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ on an interval I, y_1 and y_2 be linearly independent solutions of the DE. Initial conditions are $Y(t) = k_1$ and $Y'(t) = k_2$.

General Solution (2/2)

To solve for $(c_1, c_2)^T$, we have:

$$\begin{cases} c_1 y_1(t) + c_2 y_2(t) = k_1 \\ c_1 {y'}_1(t) + c_2 {y'}_2(t) = k_2 \end{cases} \quad \text{or} \quad \begin{pmatrix} y_1(t) & y_2(t) \\ {y'}_1(t) & {y'}_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Since the Wronskian

$$W = \begin{vmatrix} y_1(t) & y_2(t) \\ {y'}_1(t) & {y'}_2(t) \end{vmatrix} \neq 0,$$

given any k_1, k_2, there is always a unique solution for c_1, c_2. #

Example: Linear Combo. of Solutions

- $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. Are they linearly independent? By observation? By Wronskian?

- Is $y = 4\sinh 3x - 5e^{-3x}$ a solution of $y'' - 9y = 0$?

Nonhomogeneous Solutions (1/2)

- **Theorem**: Let y_p be any particular solution of the non-homogeneous linear n^{th}-order DE

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions. Then the general solution of the equation on the interval is:

$$y=c_1y_1(x)+c_2y_2(x)+\ldots+c_ny_n(x)+y_p,$$

where c_i, $i=1,2,\ldots,n$ are arbitrary constants.

Nonhomogeneous Solutions (2/2)

Proof:

Let $Y(x)$ and $y_p(x)$ be particular solutions of $L(y) = g(x)$.
Define $u(x) = Y(x) - y_p(x)$, we have

$$L(u) = L\{Y(x) - y_p(x)\} = L(Y(x)) - L(y_p(x)) = g(x) - g(x) = 0$$

Thus, $u(x)$ must be a solution to the homogeneous DE.
Therefore, $u(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$

$\rightarrow Y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$

$\rightarrow Y(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) + y_p(x)$

Any particular solution can be represented in this form.

#

Complementary Function

- The general solution of a homogeneous linear n^{th}-order DE is called the **complementary function** for the associated non-homogeneous DE.

  Let $y_c(x) = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$, the general solution of a nonhomogeneous linear n^{th}-order DE has the form:

$$y(x) = y_c(x) + y_p(x).$$

Superposition Principle for DE

- **Theorem**: Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the non-homogeneous linear n^{th}-order DE on I, corresponding to k distinct functions $g_1, g_2, \ldots, g_k$. Then,

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \ldots + y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \ldots + g_k(x).$$

Example of Superposition Principle

- Verify:

$$y_{p_1} = -4x^2 \rightarrow y'' - 3y' + 4y = -16x^2 + 24x - 8$$
$$y_{p_2} = e^{2x} \quad \rightarrow y'' - 3y' + 4y = 2e^{2x}$$
$$y_{p_3} = xe^x \quad \rightarrow y'' - 3y' + 4y = 2xe^x - e^x$$

Therefore

$$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$$

is a solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^x - e^x}_{g_3(x)}$$

Reduction of Order

- For a 2nd order linear DE, one can construct a 2nd solution y_2 from a known nontrivial solution y_1. If y_1 and y_2 are linearly independent, we must have

  $$y_2/y_1 \neq \text{constant},$$

  Therefore, $y_2(x) = u(x)y_1(x)$. Substitute this into the DE and solve for $u(x)$ is called reduction of order.

Example: $y'' - y = 0$, $y_1(x) = e^x$, find y_2

- Solution:
  Given $y_1(x) = e^x$, let $y_2(x) = u(x)\, e^x$,
  $\rightarrow y' = ue^x + e^x u',\ y'' = ue^x + 2e^x u' + e^x u''$
  $\rightarrow y'' - y = e^x(u'' + 2u') = 0$
  $\rightarrow u'' + 2u' = 0$

  Let $w = u'$, the DE becomes $w' + 2w = 0$. Multiplying by the integrating factor e^{2x}, we have $d[e^{2x}w]/dx = 0$.
  Therefore, $w = c_1 e^{-2x}$ or $u' = c_1 e^{-2x}$.
  $\rightarrow u = (-1/2)\, c_1 e^{-2x} + c_2$.
  $\rightarrow y_2(x) = u(x)\, e^x = (-c_1/2)\, e^{-x} + c_2 e^x$, let $c_1 = -2,\ c_2 = 0.$
  $\rightarrow$ Check $W(e^x, e^{-x}) \neq 0$

Solution by Reduction of Order (1/2)

- Put the 2nd order DE into the standard form:

$$y'' + P(x)y' + Q(x)y = 0,$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. If y_1 is a solution on I and that $y_1(x) \neq 0$ for all $x \in I$, by defining $y_2 = u(x)y_1$, we have:

$$y_2'' + Py_2' + Qy_2 =$$

$$u[y''_1 + Py'_1 + Qy_1] + y_1u'' + (2y'_1 + Py_1)u' = 0.$$

$$\rightarrow y_1u'' + (2y'_1 + Py_1)u' = 0$$

Solution by Reduction of Order (2/2)

- Let $w = u'$, we have $y_1 w' + (2y'_1 + Py_1)w = 0$.
  Since

$$\frac{dw}{w} = -\frac{2y_1'}{y_1}dx - Pdx \;\rightarrow\; \ln|w| = -\ln|y_1^2| - \int P(x)dx + C.$$

$$\ln|wy_1^2| = -\int P(x)dx + C \;\rightarrow\; wy_1^2 = c_1 e^{-\int P(x)dx}.$$

$$y_2 = y_1 u = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx.$$

Example: $x^2y'' - 3xy' + 4y = 0$

- Since $y_1 = x^2$ is a known solution.

$$\rightarrow \quad y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x}$$

$$= x^2 \ln x$$

The general solution is $y = c_1 x^2 + c_2 x^2 \ln x$.

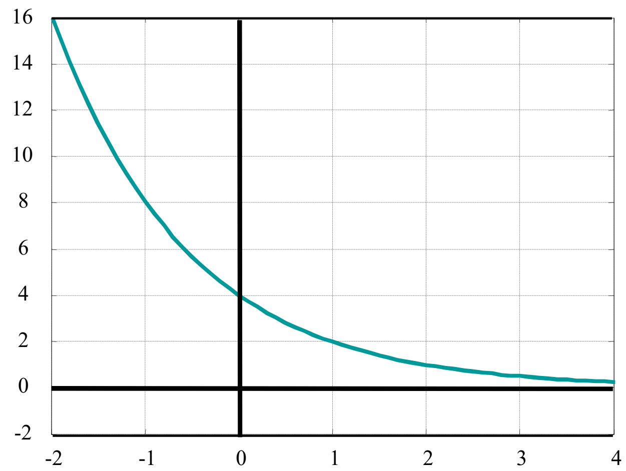
Constant Coefficients DE

- For homogeneous linear higher-order DE with real constant coefficients a_i, $i = 0, 1, \ldots, n$, $a_n \neq 0$, i.e.

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0,$$

  do we have exponential solutions?

- Recall: $by' + cy = 0$,
  $y = c_1 e^{-ax}$ on $(-\infty, \infty)$.



Auxiliary Equations

- Consider a 2nd-order DE, $ay'' + by' + cy = 0$.

  Let $y = e^{mx}$, and substituting $y' = me^{mx}$ and $y'' = m^2e^{mx}$ into the DE, we have: $am^2e^{mx} + bme^{mx} + ce^{mx} = 0$.

  $e^{mx} > 0$ for $x \in R \rightarrow am^2 + bm + c = 0$.

  This is called the **auxiliary equation** of the DE.

General Solutions (1/2)

- *Case I*, $b^2 - 4ac > 0$:
  m has two real roots m_1 and m_2, and $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ form a fundamental set of solutions.
  The general solutions is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

- *Case II*, $b^2 - 4ac = 0$:
  m has one real root m_1 and $y_1 = e^{m_1 x}$. By reduction-of-order, the 2nd solution of the DE is $y_2 = xe^{m_1 x}$.
  The general solution is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}.$$

General Solutions (2/2)

- *Case III*, $b^2 - 4ac < 0$:
  m has two complex roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Similar to *Case I*, the general solution is:

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}.$$

- By proper selection of c_1 and c_2, and using Euler's formula, $e^{i\theta} = \cos\theta + i\,\sin\theta$, it can be shown that a general solution can also be represented by

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Example: $4y''+4y'+17y = 0$

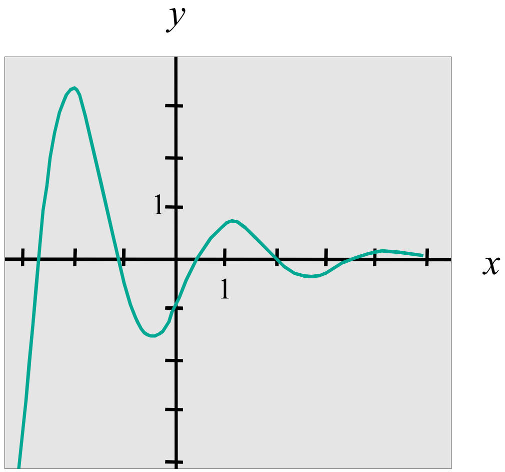
- Solve the IVP: $y(0) = -1, y'(0) = 2.$

  Solution:

  The roots of the auxiliary equation $4m^2+4m+17 = 0$ are $m_1 = -\frac{1}{2} + 2i$ and $m_2 = -\frac{1}{2} - 2i$

  $\rightarrow y = e^{-x/2} (c_1\cos 2x + c_2\sin 2x)$, with $y(0) = -1, y'(0) = 2$

  $\rightarrow y = e^{-x/2} (-\cos 2x + \frac{3}{4} \sin 2x)$



$y \rightarrow 0$, as $x \rightarrow \infty$.

Higher-Order Auxiliary Equations

- In general, to solve

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = 0,$$

where $a_i \in R$ and $a_n \neq 0$, we must solve

$$a_n m^n + a_{n-1} m^{n-1} + \ldots + a_2 m^2 + a_1 m + a_0 = 0.$$

The general solution of the DE is:

Case I (no repeated roots):

$$y = c_1 e^{m_0 x} + c_2 e^{m_1 x} + \ldots + c_n e^{m_{n-1} x}.$$

Case II (with repeated roots):

$$y = \underbrace{c_1 e^{m_0 x} + c_2 x e^{m_0 x} + \ldots + c_k x^{k-1} e^{m_0 x}}_{\text{solution form of repeated roots}} + \underbrace{c_{k+1} e^{m_1 x} + \ldots + c_n e^{m_{n-k} x}}_{\text{solution form of distinct roots}}.$$

Solution of Repeated Roots (1/2)

- For an n^{th}-order linear DE, assuming that the auxiliary equation of

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

has k repeated roots m_0. This means that the DE can be expressed as:

$$(D - m_0)^k (D - m_1) \ldots (D - m_{n-k}) y = 0.$$

Hence, the solution of $(D - m_0)^k y = 0$ will also be a solution of the nth-order DE.

Solution of Repeated Roots (2/2)

- Since $y_1 = e^{m_0 x}$ is a solution of $(D - m_0)^k y = 0$, let

$$y(x) = u(x)e^{m_0 x}.$$

Note that

$$(D - m_0)[u(x)e^{m_0 x}] = (Du(x))e^{m_0 x}.$$

Applying the operator k times on $y(x)$, we have

$$(D - m_0)^k[u(x)e^{m_0 x}] = (D^k u(x))e^{m_0 x} \text{ for any } u(x).$$

Then, $u(x)e^{m_0 x}$ is a solution of the DE $\leftrightarrow D^k u(x) = 0$.

Possible $u(x)$ that meets this condition is a polynomial with degree less than k.

$\rightarrow y(x) = (c_1 + c_2 x + \ldots + c_k x^{k-1})e^{m_0 x}$ is a family of solutions.

Non-homogeneous Linear DE

- To solve a non-homogeneous linear DE

$$a_n y^{(n)} + a_{n\text{-}1} y^{(n-1)} + \ldots + a_2 y'' + a_1 y' + a_0 y = g(x),$$

  we must do two things:

  (1) Find the complementary function y_c;
  (2) Find any particular solution y_p of the DE.
  Two methods:
  ✓ Method of undetermined coefficients
  ✓ Variation of parameters

Undetermined Coefficients (1/2)

- The method of undetermined coefficients can be applied under two conditions:

  1. a_i, $i = 0, 1, \ldots, n$, are constants, and
  2. $g(x)$ is a linear combination of functions of the following types:

$$P(x) = p_n x^n + p_{n-1} x^{n-1} + \ldots + p_2 x^2 + p_1 x + p_0,$$
$$P(x) e^{\alpha x},$$
$$P(x) e^{\alpha x} \sin \beta x,$$
$$P(x) e^{\alpha x} \cos \beta x.$$

Undetermined Coefficients (2/2)

- There are two approaches to find the particular solution given $g(x)$ using the undetermined coefficients principle:
  - Superposition approach (section 4.4 in the textbook)
    - Assume that $y_p(x)$ has similar form as $g(x)$ with some coefficients to be determined
  - Annihilator approach (section 4.5 in the textbook)
    - Try to find a linear operator L_A such that when applied to both side of the DE turns it into a higher-order homogeneous DE. That is:

$$L(y) = g(x) \;\rightarrow\; L_A \cdot L(y) = L_A \cdot g(x) = 0.$$

The extra solution subspace of $L_A \cdot L(y) = 0$ should be the subspace of the particular solution.

Example: $y'' + 4y' - 2y = 2x^2 - 3x + 6$

- By guessing, let $y_p = Ax^2 + Bx + C$, we have $y_p' = 2Ax + B$, and $y_p'' = 2A$.

  Therefore:

  $$y_p'' + 4y_p' - 2y_p$$
  $$= 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$$
  $$= -2Ax^2 + (8A - 2B)x + (2A + 4B - 2C)$$
  $$= 2x^2 - 3x + 6.$$
  $$\rightarrow y_p = -x^2 - (5/2)x - 9.$$

Example: $y'' - y' + y = 2 \sin 3x$

- By guessing, let $y_p = A \cos 3x + B \sin 3x$, we have

  $y_p' = -3A \sin 3x + 3B \cos 3x$, and

  $y_p'' = -9A \cos 3x - 9B \sin 3x$.

  Therefore:

  $$y_p'' - y_p' + y_p$$
  $$= (-9A - 3B + A) \cos 3x + (-9B + 3A + B) \sin 3x$$
  $$= 2 \sin 3x.$$
  $$\rightarrow y_p = (6/73) \cos 3x - (16/73) \sin 3x.$$

Example: y_p by Superposition

- Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.
  By super position principle, we divide the problem into two sub-problems, that is,

  $$g(x) = g_1(x) + g_2(x),$$

  where $g_1(x) = 4x - 5$, and $g_2(x) = 6xe^{2x}$.

  By guessing, let $y_{p_1} = Ax + B$, and $y_{p_2} = Cxe^{2x} + Ee^{2x}$.
  Substitute $y_p = Ax + B + Cxe^{2x} + Ee^{2x}$ into the DE, we have:

  $$y_p = -(4/3)x + (23/9) - 2xe^{2x} - (4/3)e^{2x}$$

Example: A Glitch in the Method

- Solve $y'' - 5y' + 4y = 8e^x$.
  Simply guessing that $y_p = Ae^x$ and substituting y_p into the DE gives us $0 = 8e^x$. What went wrong?

  If the guessed form of y_p falls in the solution space of y_c (i.e., $y_c = c_1e^x + c_2e^{4x}$), then we always get $0 = g(x)$.

  Solution, let $y_p = Axe^x$. Since the derivatives of y_p contains both the term Ae^x and Axe^x, it is a reasonable guess for a particular solution.

Summary of Two Cases (1/2)

- *Case I*:
  No functions in the assumed particular solution is a solution of the associated homogeneous DE.
  → Substitute with y_p = "the form of $g(x)$".

$g(x)$	y_p
1. 1 (any constant)	A
2. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
3. $\sin 4x$, or $\cos 4x$	$A \cos 4x + B \sin 4x$
4. e^{5x}	Ae^{5x}
5. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
6. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
7. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
8. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$

Summary of Two Cases (2/2)

- *Case II*:
  A function in the assumed particular solution is also a solution of the associated homogeneous DE.

  $\rightarrow$ Substitute with $y_p = x^n \times$ "the form of $g(x)$", where n is the smallest positive integer so that y_p is not in the solution space of y_c.

Examples:

- Case I
  - $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$
  - $y'' + 4y = x \cos x$
  - $y'' - 9y' + 14y = 3x^2 - 5 \sin 2x + 7xe^{6x}$

- Case II
  - $y'' - 2y' + y = e^x$
  - $y'' + y = 4x + 10 \sin x, \;\; y(\pi) = 0, y'(\pi) = 2$
  - $y'' - 6y' + 9y = 6x^2 + 2 - 12\, e^{3x}$

Annihilator Approach

- The differential operators that annihilate different $g(x)$ are as follows:
  - D^n annihilates $1, x, x^2, \ldots, x^{n-1}$.
  - $(D-\alpha)^n$ annihilates $e^{\alpha x}, xe^{\alpha x}, x^2e^{\alpha x}, \ldots, x^{n-1}e^{\alpha x}$.
  - $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$ annihilates $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$, $xe^{\alpha x}\cos\beta x$, $xe^{\alpha x}\sin\beta x$, … , $x^{n-1}e^{\alpha x}\cos\beta x$, $x^{n-1}e^{\alpha x}\sin\beta x$.

- Complementary solution to the annihilator DE gives you the form of $y_p \rightarrow$ you still need to substitute the solution form to determine the coefficients!

Example of Annihilator Approach

- Determine the y_p form of the DE: $y'' + 3y' + 2y = 4x^2$.

  The annihilator of $4x^2$ is D^3. Thus, the root of the auxiliary equation of $D^3(y) = 0$ is $m = 0, 0, 0$. The complementary solution is $y = c_1 + c_2x + c_3x^2$. Therefore, the particular solution should have the form:

$$y_p = A + Bx + Cx^2.$$

- One advantage of the annihilator approach is that the y_c of $L_A(y) = 0$ and $L(y) = 0$ can be considered jointly to choose a y_p without glitch.

Variation of Parameters (1/3)

- To adopt the variation of parameters to a linear 2nd-order DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$, one must put the DE in the standard form:

$$y'' + P(x)y' + Q(x)y = f(x).$$

We seek a particular solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous DE.

Variation of Parameters (2/3)

- Take the derivatives $y_p{}'$ and $y_p{}''$, and substitute them into the DE, we have

$$
\begin{aligned}
&y_p'' + P(x)y_p' + Q(x)y_p \\
&\qquad = u_1\left[y_1'' + P(x)y_1' + Q(x)y_1\right] + u_2\left[y_2'' + P(x)y_2' + Q(x)y_2\right] \\
&\qquad\quad + y_1u_1'' + u_1'y_1 + y_2u_2'' + u_2'y_2' + P(x)\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' \\
&\qquad = \frac{d}{dx}\left[y_1u_1'\right] + \frac{d}{dx}\left[y_2u_2'\right] + P(x)\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' \\
&\qquad = \frac{d}{dx}\left[y_1u_1' + y_2u_2'\right] + P(x)\left[y_1u_1' + y_2u_2'\right] + y_1'u_1' + y_2'u_2' = f(x).
\end{aligned}
$$

If $y_1u_1{}' + y_2u_2{}' = h(x)$, then $\begin{cases} y_1u_1' + y_2u_2' = h(x) \\ y_1'u_1' + y_2'u_2' = f(x) - h'(x) - P(x)h(x) \end{cases}$

Variation of Parameters (3/3)

- If we let $h(x) = 0$, then the solution of the system is

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

Summary of the Method

- To solve $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$:
  - Find $y_c = c_1y_1 + c_2y_2$.
  - Compute the Wronskian $W(y_1(x), y_2(x))$.
  - Put the DE into standard form: $y'' + P(x)y' + Q(x)y = f(x)$.
  - Find u_1 and u_2 by integrating $u_1' = W_1/W$ and $u_2' = W_2/W$.
  - A particular solution is $y_p = u_1y_1 + u_2y_2$.
  - The general solution is $y = y_c + y_p$.

- Note that there is no need to introduce any constants when computing the indefinite integrals of u_1' and u_2'.

Examples:

- Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.
- Solve $4y'' + 36y = \csc 3x$.
- Solve $y'' - y = 1/x$.

Higher-Order Equations

- For a linear nth-order DE

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = f(x),$$

if $y_c = c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is the complementary function of the DE, then a particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x),$$

where $u_k' = W_k/W$, $k = 1, 2, \ldots, n$ and W is the Wronskian of $y_1, y_2, .., y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column $(0, 0, \ldots, f(x))^T$.

Cauchy-Euler Equation

- Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

  where the coefficients a_i are constants, is called a Cauchy-Euler equation.

- Note that $a_n x^n = 0$ at $x = 0$. Therefore, we focus on solving the equation on $(0, \infty)$.

Method of Solution

- Assume that $y = x^m$ is a solution, we have

$$\frac{dy}{dx} = mx^{m-1}$$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$\ldots$$

$$\to a_k x^k \frac{d^k y}{dx^k}$$

$$= a_k x^k m(m-1)(m-2)\ldots(m-k+1)x^{m-k}$$

$$= a_k m(m-1)(m-2)\ldots(m-k+1)x^m.$$

2nd-Order Cauchy-Euler Eq.

- For the 2nd-order homogeneous equation:

  $$a_2x^2y'' + bxy' + cy = 0,$$

  substituting $y = x^m$ leads to

  $$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = (am(m-1) + bm + c)x^m.$$

  Thus $y = x^m$ is a solution of the DE whenever m is a solution of the auxiliary equation

  $$am(m-1) + bm + c = 0.$$

Auxiliary Equation Solutions (1/2)

- *Case I*, distinct real roots $m_1 \neq m_2$:
  Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. The general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

- *Case II*, repeated real roots $m_1 = m_2$:
  Then $y_1 = x^{m_1}$, by reduction-of-order, the 2nd solution of the DE is $y_2 = x^{m_1} \ln x$. The general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x.$$

Auxiliary Equation Solutions (2/2)

- *Case III*, conjugate complex roots:
  If $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the general solutions is

$$y = c_1 x^{(\alpha + i\beta)} + c_2 x^{(\alpha - i\beta)}.$$

- By proper selection of c_1 and c_2, and using Euler's formula, it can be shown that a general solution can also be represented by

$$y = x^{\alpha}(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)).$$

Example: Particular Solutions

- The method of undetermined coefficients does not in general carry over to variable-coefficient DEs.

  Therefore, the variation of parameters method should be used for solving non-homogeneous Cauchy-Euler equations.

- Example: Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

Reduction to Constant Coefficient Eqs

- A Cauchy-Euler equation can be reduced to a constant coefficient equation by the substitution $x = e^t$.

  Note that $dy/dt = dy/dx \cdot dx/dt = y' e^t$ and $d^2y/dt^2 = y'' e^{2t} + y' e^t$. Thus, $ax^2y'' + bxy' + cy = 0$ can be reduced to

$$ae^{2t}\left[e^{-2t}\left(\frac{d^2y}{dt^2} - y'e^t\right)\right] + be^t\left(e^{-t}\frac{dy}{dt}\right) + cy = a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0.$$

  The constant coefficient technique can be used to solve $y(t)$ and then $y(x)$ in turn.

Nonlinear Equations† (1/2)

- Nonlinear DEs do not possess superposition property.

- For example, $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \cos x$, $y_4 = \sin x$ are four linearly independent solutions of the nonlinear 2nd-order DE $(y'')^2 - y^2 = 0$ on the interval $(-\infty, \infty)$. However, the following linear combinations are not solutions:
  - $y = c_1 e^x + c_3 \cos x$
  - $y = c_2 e^{-x} + c_4 \sin x$
  - $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

† Section 4.10 of the textbook.

Nonlinear Equations (2/2)

- We could find the one-parameter family of solutions of *a few* non-linear DEs, but these solutions are not general solutions of the DEs.
- Higher order nonlinear DEs *usually* can not be solved analytically.
- Realistic physical models are often nonlinear.

Reduction of Order

- Nonlinear 2nd-order DEs of the forms
  - $F(x, y', y'') = 0$
  - $F(y, y', y'') = 0$

  can be reduced to 1st-order DEs by letting $u = y'$.

- For $F(y, y', y'') = 0$, we have $F(y, u, u') = 0$.

- For $F(y, y', y'') = 0$, observe that

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}.$$

  So the problem becomes $F(y, u, u \cdot du/dy) = 0$.

Example: y missing

- Solve $y'' = 2x(y')^2$

  Solution:
  Let $u = y'$, $du/dx = y''$, we have $du/dx = 2xu^2$

  $\rightarrow (1/u^2)\, du = 2x\, dx \quad \rightarrow \int u^{-2} du = \int 2x\, dx$
  $\rightarrow -u^{-1} = x^2 + c_1 \qquad \rightarrow -(y')^{-1} = x^2 + c_1$
  $\rightarrow dy/dx = -(x^2 + c_1)^{-1}$
  $\rightarrow y = -\int (x^2 + c_1)^{-1}\, dx$

$$\therefore\ y = -\frac{1}{\sqrt{c_1}} \tan^{-1} \frac{x}{\sqrt{c_1}} + c_2.$$

Example: x missing

- Solve $yy'' = (y')^2$

  Solution:
  Let $u = y'$, $y'' = u\, du/dy$, we have

$$y\left(u\frac{du}{dy}\right) = u^2 \;\rightarrow\; \frac{du}{u} = \frac{dy}{y}.$$

$$\rightarrow \ln|u| = \ln|y| + c_1 \quad \rightarrow u = c_2 y$$

$$\rightarrow \int (1/y)\, dy = c_2 \int dx$$

$$\rightarrow y = c_3 e^{c_2 x}.$$

Example: Taylor Series Solution (1/2)

- Let us assume that a solution of the IVP exists:
  $y'' = x + y - y^2, y(0) = -1, y'(0) = 1.$

  If $y(x)$ is analytic at 0, we have the following Taylor series expansion centered at 0:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

  Note that

$$y''(0) = 0 + y(0) - y(0)^2 = 0 + (-1) - (-1)^2 = -2.$$

Example: Taylor Series Solution (2/2)

For higher order derivatives, we have:

$$y'''(x) = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2yy',$$

$$y^{(4)}(x) = \frac{d}{dx}(1 + y' - 2yy') = y'' - 2yy'' - 2(y')^2, ...$$

and so on.

Therefore, we have:

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \cdots$$