Exercise 6.1

30. Solution: $\sum_{k=0}^{\infty}\left[k(k+2)c_k + 2(k+1)(k+2)c_{k+2}\right]x^k$.

37. Solution: $y = c_0\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{x^2}{2})^n$.

Exercise 6.2

10. Solution: $y_1 = 1 - x^2$ and $y_2 = x + \sum_{n=1}^{\infty}\frac{-1\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{(2n+1)!}x^{2n+1}$.

22. Solution: $y = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$.

24. Hint: you can let $c_0 = 1$ and $c_1 = 0$ first to solve for the first particular solution, then let $c_0 = 0$ and $c_1 = 1$ to solve for the second solution.
Solution: $y_1 = 1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots$ and $y_2 = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \cdots$.

Exercise 6.3

16. Hint:

The two lowest degrees of x are as follows:

$$[2r(r-1)+5r]c_0x^{r-1} + [2r(r+1)+5(r+1)]c_1x^r = 0.$$

If we assume $c_0 \neq 0$, then $r = 0$ or $-3/2$, and $c_1 = 0$.

For $r = 0$, we have $c_n = -\frac{c_{n-2}}{n(2n+3)}$, $n = 2,3,4,\ldots$.

For $r = -3/2$, we have $c_n = -\frac{c_{n-2}}{(2n-3)n}$, $n = 2,3,4,\ldots$.

In addition, $c_1 = c_3 = c_5 = \ldots = 0$.

Solution: $y = a_0\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{2^k k![7\cdot 11\cdots(4k+3)]}x^{2k}\right] + a_1x^{-3/2}\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{2^k k![1\cdot 5\cdots 4k-3]}x^{2k}\right]$.

32. Hint: For $r_2 = 0$, we have $-(n-4)c_n + (n-3)c_{n-1} = 0$, $n = 1, 2, 3, \ldots$.
Solution: $y = c_0\left(1 + \frac{2}{3}x + \frac{1}{3}x^2\right) + c_4\sum_{n=4}^{\infty}(n-3)x^n$.

Exercise 6.4

11. Hint: The Bessel's equation is $x^2v'' + xv' + (\alpha^2x^2 - 1/4)v = 0$.

Solution: $y = c_1 x^{-\frac{1}{2}} J_{\frac{1}{2}}(\alpha x) + c_2 x^{-\frac{1}{2}} J_{-\frac{1}{2}}(\alpha x)$.

35. Hint: $\dfrac{dx}{dt} = \dfrac{dx}{ds}\dfrac{ds}{dt} = \dfrac{dx}{ds}\left(-\sqrt{\dfrac{k}{m}}e^{-\alpha t/2}\right)$, $\dfrac{d^2x}{dt^2} = \dfrac{dx}{ds}\left(\dfrac{\alpha}{2}\sqrt{\dfrac{k}{m}}e^{-\alpha t/2}\right) + \dfrac{d^2x}{ds^2}\left(\dfrac{k}{m}e^{-\alpha t}\right)$.