Differential Equations '19 – Midterm

11/4/2019

Note: 20 points for each problem. Partial grades will be given even for incomplete solutions so please do not leave blanks.

1. Solve the following initial value problem: $xy' = (x^3e^x + \ln(y))\, y$, $y(1) = 1$. (*Hint: try the method of substitution by $u = \ln y$*).

[Solution]

The D.E. can be written as: $x\dfrac{y'}{y} = x^3e^x + \ln y$, $y \neq 0$.

Let $u = \ln y$, we have $\dfrac{du}{dx} = \dfrac{du}{dy}\cdot\dfrac{dy}{dx} = \dfrac{y'}{y} \rightarrow \dfrac{du}{dx} - \dfrac{1}{x}u = x^2e^x$.

The integrating factor is $e^{-\int dx/x} = \dfrac{1}{x}$.

$\dfrac{d}{dx}\left[\dfrac{1}{x}u\right] = \dfrac{1}{x}\cdot x^2e^x = xe^x \rightarrow \dfrac{u}{x} = (x-1)e^x + c$.

$\ln y = x(x-1)e^x + cx$, $y(1) = 1 \rightarrow c = 0$.

Therefore, the solution to the IVP is $\ln y = x(x-1)e^x$ or $y = e^{x(x-1)e^x}$. #

2. Solve the initial value problem: $3y^2y' - (xy' + y)\sin(xy) + 2x = 0$, $y(0) = 2$.

[Solution]

The differential equation can be written as $(2x - y\sin xy)dx + (3y^2 - x\sin xy)\,dy = 0$.

Thus, $M(x, y) = 2x - y\sin xy$, and $N(x, y) = 3y^2 - x\sin xy$.

Since $\dfrac{\partial M}{\partial y} = -\sin xy - xy\cos xy = \dfrac{\partial N}{\partial x}$, the D.E. is an exact equation.

$f(x, y) = \int M(x, y)dx + g(y) \rightarrow f(x, y) = x^2 + \cos xy + g(y)$.

$N(x, y) = f_y(x, y) = -x\sin xy + g'(y) \rightarrow g'(y) = N(x, y) + x\sin xy$,

$\rightarrow g(y) = \int(3y^2 - x\sin xy + x\sin xy)dy = y^3$.

Therefore, $f(x, y) = x^2 + \cos xy + y^3$ and the implicit solution is $x^2 + \cos xy + y^3 = C$.

The solution to the IVP is then $x^2 + \cos xy + y^3 = 9$ because $y(0) = 2$. #

3. Find the general solution of the differential equation $3y'' - 6y' + 6y = x + e^x \sec x$. (*Hint:* $\int \tan x\, dx = -\ln|\cos x| + C$).

[Solution]

The auxiliary equation is $3m^2 - 6m + 6 = 0$, so $y_c = e^x(c_1\cos x + c_2\sin x)$. We can use the superposition principle to divide the DE into two subsystems.

1) For $y'' - 2y' + 2y = (1/3)\,x$, use the method of undetermined coefficients to solve for the first particular solution y_{p_1}:

   $y_{p_1} = Ax + B$, $y'_{p_1} = A$, and $y''_{p_1} = 0$.

   $\rightarrow -2A + (2Ax + 2B) = (1/3)\,x$.

   $\rightarrow A = (1/6)$, $B = (1/6)$. Therefore, $y_{p_1}(x) = (1/6)\,x + (1/6)$.

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2) For $y'' - 2y' + 2y = (1/3)\, e^x \sec x$, use the variations of parameters to solve for the second particular solution y_{p_2}:

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \cos x + e^x \sin x \end{vmatrix} = e^{2x}.$$

Since $f(x) = (1/3)\, e^x \sec x$, we obtain

$$u_1' = \frac{(e^x \sin x)(e^x \sec x)/3}{e^{2x}} = -\frac{1}{3}\tan x, \qquad u_2' = \frac{(e^x \cos x)(e^x \sec x)/3}{e^{2x}} = \frac{1}{3}.$$

Then $u_1 = (1/3) \ln|\cos x|$, $u_2 = (1/3)x$, and $y_{p_2}(x) = (\ln|\cos x|\, e^x \cos x + xe^x \sin x)/3$.

The overall solution of y is: $y(x) = e^x (c_1 \cos x + c_2 \sin x) + y_{p_1}(x) + y_{p_2}(x)$. #

4. Find two linearly independent, piecewise continuous solutions $y_1(x)$ and $y_2(x)$ of the IVP:

$y'' + (\mathrm{sgn}\, x)y = 0$, $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$. Note that $\mathrm{sgn}\, x = \begin{cases} +1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$.

Please define the largest interval of definition I.

[Solution]

If $x > 0$ the D.E. is $y'' + y = 0$, the general solution is $y = A \cos x + B \sin x$.

If $x < 0$, then the D.E. becomes $y'' - y = 0$, the general solution is $y = C\, e^x + D\, e^{-x}$.

If $x = 0$, then the D.E. is $y''(0) = 0 \rightarrow y$ has curvature 0 at $x = 0$, y could be a non-polynomial function. This is different from $y'' = 0$, $\forall x$.

To satisfy the initial conditions, $y_1(0) = 1$, $y_1'(0) = 0$, we choose $A = 1$, $B = 0$, and $C = 1/2$, $D = 1/2$.

But to satisfy $y_2(0) = 0$, $y_2'(0) = 1$, we choose $A = 0$, $B = 1$, and $C = 1/2$, $D = -1/2$.

Therefore, we have

$$y_1(x) = \begin{cases} \cos x, & x \geq 0 \\ (e^x + e^{-x})/2, & < 0 \end{cases}, \quad \text{and} \quad y_2(x) = \begin{cases} \sin x, & x \geq 0 \\ (e^x - e^{-x})/2, & < 0 \end{cases}.$$

Note that the curvature for both y_1 and y_2 at $x = 0$ are zero and the curve is continuous at 0.

Therefore, the largest interval of definition that fulfills both initial conditions is $(-\infty, \infty)$. #

5. For a damped spring-mass system, the vertical offset x from its equilibrium position can be modeled by a 2nd-order differential equation $x'' + 2\lambda\, x' + x = f(t)$, $x(0) = x'(0) = 0$, where the external force is observed as $f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 0, & t > 1 \end{cases}$.

If the system is a critically damped system, answer the following questions:

(a) What is the unique solution $x(t)$ of the IVP for $t \geq 1$?

(b) At what value would the system pass through the equilibrium position for $t \in (0, \infty)$?

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[Solution]

A critically damped system has the DE: $x'' + 2\lambda x' + \omega_0^2 x = f(t)$, where $\lambda^2 = \omega_0^2$.

In this problem, $\lambda = \pm 1$, we can set $\lambda = 1$ (positive natural frequency), the derivation for $\lambda = -1$ is similar.

Therefore, the system equation is:

$$\begin{cases} x'' + 2x' + x = t, & 0 \le t \le 1 \\ x'' + 2x' + x = 0, & t > 1 \end{cases}.$$

Solving the auxiliary equation, the complementary solution is $x_c(t) = e^{-t}(c_1 + c_2 t)$.

By method of undetermined coefficients, $x_p(t) = t - 2$, $0 \le t \le 1$.

The general solution of the D.E. is $x(t) = e^{-t}(c_1 + c_2 t) + (t - 2)$, $0 \le t \le 1$.

Since $x(0) = x'(0) = 0$, we have the particular solution of the IVP as:

$$x(t) = e^{-t}(2 + t) + (t - 2),\ 0 \le t \le 1. \qquad (1)$$

For the general solution when $t > 1$, we must first determine find $x(1)$ and $x'(1)$.

At $t = 1$, from equation (1), we have $x(1) = 3e^{-1} - 1$, $x'(1) = 1 - 2e^{-1}$.

For $t > 1$, the equation becomes an IVP problem of a homogeneous equation:

$x(t) = e^{-t}(c_1 + c_2 t)$, $x(1) = 3e^{-1} - 1$, $x'(1) = 1 - 2e^{-1}$.

(a) Solving for c_1 and c_2, we have the particular solution:

$x(t) = e^{-t}(2 - e + t)$, for $t > 1$.

(b) Since $e^{-t} > 0$, $(2 - e + t) = 0$ only when $t = e - 2 < 1$,

this system will never pass through $x = 0$ for $t \in (0, \infty)$.

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