Differential Equations 2019 – Final Exam Solutions.

1. (a) Use Laplace transform to solve the IVP: $x'' + 4x' + 5x = \delta(t-\pi) + \delta(t-2\pi)$, $x(0) = 0$, $x'(0) = 2$.
   (b) Use Laplace transform to find a nontrivial family of solution of $tx'' - 2x' + tx = 0$, $x(0) = 0$.
   Hint: $\sin A \sin B = [\cos(A-B) - \cos(A+B)]/2$.

   [Solution]

   (a) Transform the D.E. into Laplace domain:

   $[s^2X(s) - sx(0) - x'(0)] + 4sX(s) - x(0) + 5X(s) = e^{-\pi s} + e^{-2\pi s}$

   $[s^2X(s) - 2] + 4sX(s) + 5X(s) = e^{-\pi s} + e^{-2\pi s}$

   $$X(s) = \frac{2 + e^{-\pi s} + e^{-2\pi s}}{(s+2)^2 + 1}$$

   $$X(s) = \left.\frac{2}{s^2+1}\right|_{s\to s+2} + \left.\frac{e^{-\pi(s-2)}}{s^2+1}\right|_{s\to s+2} + \left.\frac{e^{-2\pi(s-2)}}{s^2+1}\right|_{s\to s+2}$$

   Since $\mathcal{L}^{-1}\left\{\left.\frac{e^{-\pi(s-2)}}{s^2+1}\right|_{s\to s+2}\right\} = e^{2\pi}\cdot \mathcal{L}^{-1}\left\{\left.\frac{e^{-\pi s}}{s^2+1}\right|_{s\to s+2}\right\} = e^{2\pi} e^{-2t}\sin(t-\pi)u(t-\pi)$, we have

   $x(t) = 2e^{-2t}\sin(t) + e^{-2(t-\pi)}\sin(t-\pi)u(t-\pi) + e^{-2(t-2\pi)}\sin(t-2\pi)u(t-2\pi)$

   $\quad = [2 - e^{2\pi}u(t-\pi) + e^{4\pi}u(t-2\pi)]\, e^{-2t}\sin t.$ #

   (b) Transform the D.E. into Laplace domain:

   $$-\frac{d}{ds}[s^2X(s) - sx(0) - x'(0)] - 2[sX(s) - x(0)] - \frac{d}{ds}X(s) = 0.$$

   $-[2sX(s) + s^2X'(s)] - 2sX(s) - X'(s) = 0.$

   $(s^2+1)X'(s) + 4\,sX(s) = 0.$

   $\frac{dX(s)}{X(s)} = -\frac{4s}{s^2+1}ds \;\to\; \ln|X(s)| = -2\ln|s^2+1|.$

   $X(s) = c_1/(s^2+1)^2$, $c_1 \neq 0 \;\to\; x(t) = c_1\mathcal{L}^{-1}\{1/(s^2+1)^2\} = c_1\int_0^t \sin(\tau)\sin(t-\tau)\,d\tau.$

   $x(t) = c_1\int_0^t \frac{1}{2}[\cos(2\tau - t) - \cos(t)]d\tau = c\,(\sin(t) - t\cos(t))$, $c \neq 0.$ #

2. Find the fundamental matrix $\Phi(t)$ of the system $\begin{cases} x'_1 = x_2 + x_3 \\ x'_2 = x_1 + x_3 \\ x'_3 = x_1 + x_2 \end{cases}$ with the initial condition $\mathbf{\Phi}(0) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. *Hint: the characteristic equation is* $(\lambda+1)^2(\lambda-2) = 0$.

   [Solution]

   The system matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and the characteristic eq. is $(\lambda+1)^2(\lambda-2) = 0$.

   For $\lambda = -1$, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Two eigen vectors are $(1, 0, -1)^T$ and $(0, 1, -1)^T$.

   For $\lambda = 2$, $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. One eigen vector is $(1, 1, 1)^T$.

Therefore, a fundamental matrix of the system is:

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 & e^{2t} \\ 0 & e^{-t} & e^{2t} \\ -e^{-t} & -e^{-t} & e^{2t} \end{pmatrix} \text{ and we have } \Psi(0) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \neq \mathbf{I}.$$

This fundamental matrix does not satisfy the initial condition $\mathbf{\Phi}(0) = \mathbf{I}$. However, any fundamental matrix is related to $\mathbf{\Psi}$ by $\mathbf{\Phi} = \mathbf{\Psi C}$, where $\mathbf{C}$ is some constant matrix.
We must solve for the $\mathbf{C}$ matrix such that $\mathbf{\Phi}(0) = \mathbf{\Psi}(0)\mathbf{C} = \mathrm{I}$. That is $\mathbf{C} = \mathbf{\Psi}^{-1}(0)$.

Since $\mathbf{C} = \mathbf{\Psi}^{-1}(0) = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$, and $\mathbf{\Phi} = \mathbf{\Psi C}$, $\Phi(t) =$

$$\frac{1}{3}\begin{pmatrix} 2e^{-t} + e^{2t} & -e^{t} + e^{2t} & -e^{t} + e^{2t} \\ -e^{t} + e^{2t} & 2e^{-t} + e^{2t} & -e^{t} + e^{2t} \\ -e^{t} + e^{2t} & -e^{t} + e^{2t} & 2e^{-t} + e^{2t} \end{pmatrix}.$$

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3. Find the general solution of the differential equation $2x^2y'' - xy' + (1 + x)y = 0$ using the Frobenius' method. Note: you must solve the recurrence relations to express c_n as a function of n.

[Solution]

Let $y = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$,
$2x^2y'' - xy' + (1+x)y =$

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r-1} - \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1}$$

$= c_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty}\{[2(n+r)(n+r-1) - (n+r) + 1]c_n + c_{n-1}\}x^{n+r} = 0.$

The indicial equation is $(r-1)(2r-1) = 0 \rightarrow r = 1, 1/2$.

$$c_n = -\frac{c_{n-1}}{2(n+r)^2 - 3(n+r) + 1} = -\frac{c_{n-1}}{[(n+r)-1][2(n+r)-1]}, \qquad n \geq 1.$$

For $r = 1$,

$$c_n = -\frac{c_{n-1}}{(2n+1)n} = \frac{(-1)^n}{[3 \cdot 5 \cdot 7 \cdots (2n+1)]n!}c_0, \quad n \geq 1.$$

Since $2 \cdot 4 \cdot 6 \ldots 2n = 2^n n!$, we have

$$c_n = \frac{(-1)^n 2^n}{(2n+1)!}c_0, \quad n \geq 1.$$

For $r = 1/2$,

$$c_n = -\frac{c_{n-1}}{(2n-1)n} = \frac{(-1)^n}{[1 \cdot 3 \cdot 5 \cdots (2n-1)]n!}c_0, \quad n \geq 1.$$

$$c_n = \frac{(-1)^n 2^n}{(2n)!}c_0, \quad n \geq 1.$$

Therefore, the general solution is:

$$y = C_0\left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n+1)!}x^{n+1}\right] + C_1\left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!}x^{n+\frac{1}{2}}\right]$$

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4. Find a particular solution of the equation $\frac{1}{4}\frac{d^2x}{dt^2}+12x=f(t)$ using Fourier series, where the driving force is defined by $f(t)=\begin{cases} t, & 0\le t<½ \\ 1-t, & ½<t<1\end{cases}$; $f(t+1)=f(t)$ for all $t\in R$.

[Solution]

We have

$$a_0=\frac{2}{(1/2)}\int_0^{1/2} t dt=\frac{1}{2}, \qquad a_n=\frac{2}{(1/2)}\int_0^{1/2} t\cos 2n\pi t dt=\frac{1}{n^2\pi^2}[(-1)^n-1].$$

So that

$$f(t)=\frac{1}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi^2}\cos 2n\pi t.$$

Substituting the assumption

$$x_p(t)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos 2n\pi t.$$

Into the differential equation then gives

$$\frac{1}{4}x''_p+12x_p=6A_0+\sum_{n=1}^{\infty}A_n(12-n^2\pi^2)\cos 2n\pi t=\frac{1}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2\pi^2}\cos 2n\pi t$$

and $A_0=\frac{1}{24}$, $A_n=\frac{(-1)^n-1}{n^2\pi^2(12-n^2\pi^2)}$.

Thus

$$x_p(t)=\frac{1}{48}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2(12-n^2\pi^2)}\cos 2n\pi t. \qquad \#$$

5. Derive the solution $y(x, t)$, $0\le x\le 2$, $t\ge 0$, of the following boundary value problem: $\frac{\partial^2 y}{\partial t^2}=4\frac{\partial^2 y}{\partial x^2}$, $y(0, t)=y(2, t)=0$, $y(x, 0)=(x-1)^2-1$, and $y_t(x, 0)=0$. Note: you must derive the solution using separable function assumption. You cannot use the wave equation formula to find the solution.

[Solution]

By separation of variables, substitution of $y(x, t)=X(x)T(t)$ in $y_{tt}=4y_{xx}$ yields $XT''=4X''T$ for all x and t. Therefore, assume that $\frac{X''}{X}=\frac{T''}{2^2T}=-\lambda$, for some λ.

We have a system of ODE that must satisfy $y_t(x, 0)=0$:

$$\begin{cases} X''+\lambda X=0, & X(0)=X(2)=0 \\ T''+2^2\lambda T=0, & T'(0)=0\end{cases}.$$

The first equation has non-trivial solution when $\lambda_n=n^2\pi^2/2^2$, $n=1,2,3,\dots$ and $X_n(x)=\sin(n\pi x/2)$, $n=1, 2, 3, \dots$. Substitute λ_n into the 2nd eq. we have:

$T''_n+n^2\pi^2T_n=0,\ T'_n(0)=0.$

The solution for $T_n(t)=A_n\cos(n\pi t)$, $n=1,2,3,\dots$.

Thus, we have $y(x,t)=\sum_{n=1}^{\infty}y_n(x,t)=\sum_{n=1}^{\infty}X_n(x)T_n(t)=\sum_{n=1}^{\infty}A_n\sin(\frac{n\pi x}{2})\cos(n\pi t)$.

The boundary condition says: $y(x,\ 0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{2})$= $f(x)$, $0 < x < 2$.

The Fourier transform of $f(x)$ is

$$A_n = \int_0^2 x(x-2)\sin(\frac{n\pi x}{2})\,dx$$

$$=(-\frac{2}{n\pi})x^2\cos\frac{n\pi x}{2}\Big|_0^2 - \int_0^2 2x(-\frac{2}{n\pi})\cos\frac{n\pi x}{2}\,dx - \int_0^2 2x\sin\frac{n\pi x}{2}\,dx$$

$$=\frac{(-1)^{n+1}\cdot 8}{n\pi} + \left[\frac{16}{(n\pi)^3}((-1)^n - 1)\right] - \left[\frac{(-1)^{n+1}\cdot 8}{n\pi}\right]$$

$$= \frac{16}{(n\pi)^3}((-1)^n - 1)$$

The solution is: $y(x,t) = \sum_{n=1}^{\infty}\left(\frac{16}{(n\pi)^3}((-1)^n - 1)\right)\sin(\frac{n\pi x}{2})\cos(n\pi t)$.

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