Link Probability, Network Coverage, and Related Properties of Wireless Ad Hoc Networks

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***Abstract*— This paper has analyzed link probability, expected node degree, expected number of links, and expected area collectively covered by a finite number of nodes in wireless ad hoc networks. Apart from the formulation of exact mathematical expressions for these properties, we have disclosed two fundamental results: (1) Every possible link has an equal probability of occurrence. (2) It is the border effects that makes two links probabilistically dependent. Simulation results show that our analysis predicts related measure with accuracy.**

I. Introduction

We define an $\langle n, r, l, m\rangle$-network as a wireless ad hoc network (MANET) that possesses the following properties: (1) The network consists of n nodes placed in an $l \times m$ rectangle area. (2) The position of each node is a random variable uniformly distributed over the given area. (3) Each node has a transmission radius of r unit length, where $r \leq \min(l, m)$. (4) Any two nodes that are within the transmission range of each other will have a link connecting them[1]. We are concerned with several fundamental properties in this model.

It was commonly believed that the probability of link occurrence in MANET cannot be identical. However, we found that it is not true. The expected node degree and the expected number of links in a MANET have also been obtained. Previous work on degree estimate [1], [2], [3] does not take into account *border effects* [2], which refers to the circumstance that a node placed near the system border will cover less area (with its radio signal) than nodes placed midway. Border effects makes the conventional estimate inaccurate. In contrast, our results are not subject to border effects.

The next problem to solve is the expected area jointly covered by a finite number of nodes, which is a form of so-called *coverage problem*. Given the expected node coverage, which can be derived from link probability, the problem at hand is still complicated by the fact that region covered by each node may overlap one another in a stochastic way.

We also found that border effects are not only a major obstacle to precise calculations of many network properties, but also the reason behind the probabilistic dependency of two links. This implies that the occurrences of any two links are independent to each other if border effects disappear.

[1]This is a simplified model as only path loss is taken into account. In a practical network, different nodes would experience different shadowing, thus making the transmission radius different for different nodes.

We conducted experiments for a quantitative analysis of the impacts of border effects. The numerical results show that our analysis accurately estimates these network properties.

II. Link Probability and Expected Degree

This section computes analytically the probability that two arbitrary nodes are within the transmission range of each other. Let the position of node i be determined by Cartesian coordinates (X_i, Y_i), where $0 \leq X_i \leq l$ and $0 \leq Y_i \leq m$. Clearly, X_i's are iid random variables with p.d.f. $f(x) = 1/l$ over the range $[0, l]$, while Y_i's are iid with p.d.f. $f(y) = 1/m$ over $[0, m]$.

Lemma 1: For any two distinct nodes i and j in an $\langle n, r, l, m\rangle$-network with positions (X_i, Y_i) and (X_j, Y_j), respectively, let $Z_i = |X_i - X_j|$ and $W_i = |Y_i - Y_j|$. We have $\Pr[Z_i \leq z] = (-z^2 + 2lz)/l^2$, $0 \leq z \leq l$, and $\Pr[W_i \leq w] = (-w^2 + 2mw)/m^2$, $0 \leq w \leq m$.

Proof: We show only the result for $\Pr[Z_i \leq z]$. The result for $\Pr[W_i \leq w]$ can be derived in a similar way. We know that $\Pr[Z_i \leq z] = \Pr[X_i < X_j \leq X_i + z] + \Pr[X_j < X_i \leq X_j + z]$. The value of $\Pr[X_i < X_j \leq X_i + z]$ can be calculated by taking integrals over two non-overlapping intervals and then adding them up. The first interval corresponds to when $X_i + z \leq l$. We have $\Pr[X_i < X_j \leq X_i + z \leq l] = \int_0^{l-z}\int_{x_i}^{x_i+z} f(x_i, x_j)dx_j dx_i$, where $f(x_i, x_j)$ is the joint p.d.f. of X_i and X_j. Since X_i and X_j are independent, $f(x_i, x_j) = f(x_j)f(x_j) = 1/l^2$. So $\Pr[X_i < X_j \leq X_i + z \leq l] = z(1 - z)/l^2$. The second interval corresponds to when $X_i + z > l$. We have $\Pr[l - z < X_i < X_j \leq l] = z^2/2l^2$. Therefore, $\Pr[X_i < X_j \leq X_i + z] = \frac{z}{l^2}(l - z) + \frac{z^2}{2l^2} = \frac{-z^2+2lz}{2l^2}$. Similarly, $\Pr[X_j < X_i \leq X_j + z] = \frac{-z^2+2lz}{2l^2}$. It follows that $\Pr[Z_i \leq z] = \frac{-z^2+2lz}{l^2}$. □

Lemma 2: For any two distinct nodes i and j in an $\langle n, r, l, m\rangle$-network with positions (X_i, Y_i) and (X_j, Y_j), respectively, let $U_i = (X_i - X_j)^2$ and $V_i = (Y_i - Y_j)^2$. The p.d.f. of U_i is $f(u) = (\frac{l}{\sqrt{u}} - 1)/l^2$, $0 \leq u \leq l^2$, and the p.d.f. of V_i is $g(v) = (\frac{m}{\sqrt{v}} - 1)/m^2$, $0 \leq v \leq m^2$.

Proof: Let $F(u)$ be the probability distribution function of U_i. We have $F(u) = \Pr[U_i \leq u] = \Pr[Z_i \leq \sqrt{u}]$, $0 \leq u \leq l^2$, where $Z_i = |X_i - X_j|$. By Lemma 1 we have $\Pr[Z_i \leq \sqrt{u}] = -u + 2l\sqrt{u}/l^2$. Therefore the p.d.f. of U_i is $f(u) = F'(u) = (\frac{l}{\sqrt{u}} - 1)/l^2$, $0 \leq u \leq l^2$. Similarly, the p.d.f. of V_i is $g(v) = (\frac{m}{\sqrt{v}} - 1)/m^2$, $0 \leq v \leq m^2$. □

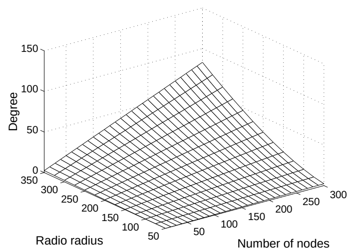

Fig. 1. Expected degree for $n = 10$ to 300 and $r = 50$ to 350 in a 1000×1000 rectangle.

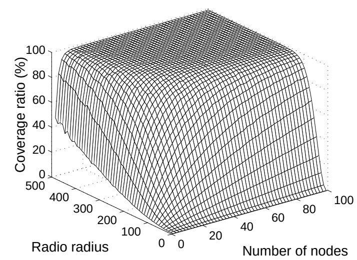

Fig. 2. Ratios of the theoretical network coverage to the whole system area, with n ranging from 1 to 99 and r ranging from 1 to 491.

Theorem 1: In an $\langle n, r, l, m\rangle$-network, the occurrence probability of link $\langle i, j\rangle$ between any two distinct nodes i and j is $(\frac{1}{2}r^4 - \frac{4}{3}lr^3 - \frac{4}{3}mr^3 + \pi r^2 ml)/m^2l^2$.

Proof: Link $\langle i, j\rangle$ forms if and only if the distance between them is not greater than r. Thus the probability of link $\langle i, j\rangle$ is $\Pr[U_i + V_i \leq r^2] = \int_0^{r^2}\int_0^{r^2-u} h(u,v)dvdu$, where $U_i = (X_i - X_j)^2$, $V_i = (Y_i - Y_j)^2$, and $h(u, v)$ is the joint p.d.f. for U_i and V_i. Since U_i and V_i are independent, we have $h(u, v) = f(u)g(v)$, where $f(u)$ and $g(v)$ are as defined in Lemma 2. It follows that $\Pr[U_i + V_i \leq r^2] = (\frac{1}{2}r^4 - \frac{4}{3}lr^3 - \frac{4}{3}mr^3 + \pi r^2 ml)/m^2l^2$. □

Theorem 1 indicates that the probability of link $\langle i, j\rangle$ depends on the values of m, l, and r but not on i, j, or n, and all links have equal probability. The result of identical link probability does not contradict the thought that link occurrences are correlated.

Given n random variables R_i, where $i = 1$ to n, it is known [4] that $E[R_1+R_2+\cdots+R_n] = E[R_1]+E[R_2]+\cdots+E[R_n]$ regardless whether R_i's are independent to each other. Since each node may have $n - 1$ links and there are potentially $n(n-1)/2$ links between n nodes, we have the following two corollaries.

Corollary 1: The average (expected) node degree in an $\langle n, r, l, m\rangle$-network is $(n - 1)(\frac{1}{2}r^4 - \frac{4}{3}lr^3 - \frac{4}{3}mr^3 + \pi r^2 ml)/m^2l^2$.

Corollary 2: The expected number of links in an $\langle n, r, l, m\rangle$-network is $n(n - 1)(\frac{1}{2}r^4 - \frac{4}{3}lr^3 - \frac{4}{3}mr^3 + \pi r^2 ml)/2m^2l^2$.

Fig. 1 shows the expected degree estimated by Corollary 1 for various n and r.

Theorem 2: In an $\langle n, r, l, m\rangle$-network with $r \leq \min(l/2, m/2)$, the expected transmission coverage area of a single node is $\phi = (\frac{1}{2}r^4 - \frac{4}{3}lr^3 - \frac{4}{3}mr^3 + \pi r^2 ml)/ml$.

Proof: It is straightforward since link probability derived in Theorem 1 is equal to ϕ/lm. The result has also been confirmed by geometric computation (for details, refer to [5]). □

III. Expected Network Coverage

Let C_n be the expected area jointly covered by n randomly placed nodes, referred to as *network coverage*. We want to express C_n in terms of expected node coverage ϕ.

The deployment of nodes can be thought of as an iterative process that places nodes one by one. Suppose $n - 1$ nodes have already been placed. When we add the nth node to the $(n - 1)$-node network, the extra coverage area contributed by this newly placed node is a portion of its node coverage. Let ρ_n denote the proportion of this portion to the node coverage. C_n can be expressed as a recurrence relation as $C_n = C_{n-1}+\rho_n\phi$. Since nodes are uniformly distributed, ρ_n is expected to be the proportion of the uncovered area to the whole target area. Thus we have $\rho_n = (A-C_{n-1})/A$, where A denotes the area of the target region. It turns out that $C_n = C_{n-1} + (1 - C_{n-1}/A)\phi$. Since $C_1 = \phi$, solving this recurrence relation yields

$$C_n = [1 - (1 - \phi/A)^n]A. \quad (1)$$

Eq. (1) holds for any shape of target region as well as for any shape of node's coverage. Let us focus on $l \times m$ rectangular where $A = lm$ and, if border effects are not taken into account, $\phi = \pi r^2$. Eq. (1) becomes

$$C_n = [1 - (1 - \pi r^2/lm)^n]lm. \quad (2)$$

This is a rough estimation for expected network coverage. The following theorem gives us a precise estimation considering border effects.

Theorem 3: For an $\langle n, r, l, m\rangle$-network with $l \geq 2r$ and $m \geq 2r$, the expected area collectively covered by all nodes is

$$C_n = \left[1 - \left(\frac{m^2l^2 - \frac{1}{2}r^4 + \frac{4}{3}lr^3 + \frac{4}{3}mr^3 - \pi r^2 ml}{m^2l^2}\right)^n\right] lm$$

.

Proof: We have $A = lm$ for an $l \times m$ rectangle. By Theorem 2 and (1), we obtain the result. □

Fig. 2 shows the ratios of the theoretical network coverage to the whole system area for various n and r.

IV. Link Dependency

Many researchers (e.g., [1]) have pointed out that link occurrences are not independent events. Their arguments are mainly based on a three-link scenario: the event that both link $\langle X, Y\rangle$ and link $\langle X, Z\rangle$ show up is not independent of the event that $\langle Y, Z\rangle$ exists. However, few studies have reported on the dependency of any two links.

Two links that share no common endpoint node are obviously independent to each other. Let X, Y, and Z be three nodes and consider L_{XY}, the event that link $\langle X, Y\rangle$ exists, and

L_{XZ}, the event that link $\langle X, Z \rangle$ exists. When X is located at (x, y), the probability that both Y and Z are located in X's coverage is $[c(x, y)/lm]^2$, where $c(x, y)$ denotes the area that a node located at (x, y) covers. Thus the joint link probability of L_{XY} and L_{XZ} is

$$\Pr[L_{XY}, L_{XZ}] = \frac{1}{lm} \int_0^l \int_0^m \left[\frac{c(x, y)}{lm} \right]^2 dydx. \qquad (3)$$

Theorem 4: If border effects can be removed but system area remains constant (which can be achieved by using, e.g., torus convention [6], [3]), the occurrences of any two links are independent to each other.

Proof: Clearly, $c(x, y) = \pi r^2$ for all x, y if border effects disappear. Thus $\Pr[L_{XY}] = \Pr[L_{XZ}] = \pi r^2/lm$. By (3), we have $\Pr[L_{XY}, L_{XZ}] = \Pr[L_{XY}] \Pr[L_{XZ}]$ for all X, Y, Z. □

Corollary 3: It is the border effects that makes any two links in an $\langle n, r, l, m \rangle$-network dependent.

Note that the three-link argument remains valid regardless of border effects.

V. Simulations and Numerical Results

We conducted additional experiments for a quantitative analysis of the impacts of border effects on network properties. The first property we measured is average degree. Fig. 3(a) shows average degrees estimated with Poisson point process [2] (the rough estimate) while Fig. 3(b) shows the results obtained from the simulation. Fig. 3(c) shows the errors of Corollary 1 in comparison with the simulated results, where the error is defined as |estimated value − measured value|/measured value. The mean is 2.56×10^{-4} while the standard deviation is 4.81×10^{-4}. Fig. 3(c) shows the errors of the rough estimate in comparison with the simulated results. Clearly, the errors are in proportional to the radio radius r (the mean is 0.22 and the standard deviation is 0.11). This can be explained as the impacts of border effects become significant as the radio radius becomes large. In contrast, the largest error of our estimate is only 0.6%, occurring on the smallest n and r.

We next measured coverage ratio, the ratio of the network coverage to the whole system area. Fig. 4(a) shows results estimated with Eq. (2). Fig. 4(b) shows the results obtained from the experiments. The errors of Theorem 3 in comparison with the simulated results are shown in Fig. 4(c), with mean $= 0.50 \times 10^{-2}$ and standard deviation $= 0.68 \times 10^{-2}$. Fig. 4(d) shows the errors of the results estimated with (2). The mean is 2.97×10^{-2} and the standard deviation is 5.78×10^{-2}. We conclude that Theorem 3 is more accurate and has smaller variance than (2).

VI. Conclusions

Exact mathematical expressions for link probability, expected node degree, expected number of links, and expected node and network coverage have been formulated. It has been shown that every possible link in a MANET has equal probability of occurrence. It is also proven that two links are probabilistically independent to each other if there is no border effect. Additional experimental results confirm our analysis.

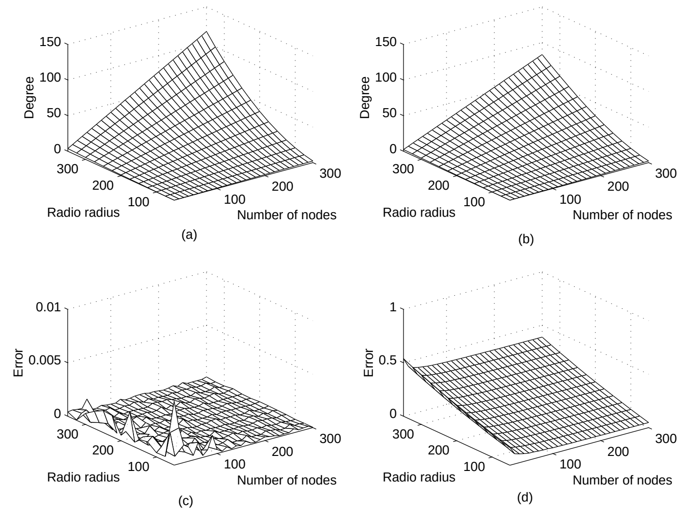

Fig. 3. Average degree in 1000×1000 rectangle. (a) Results of rough estimate. (b) Simulated results. Each value is averaged over 100,000 experiments. (c) Errors of precise estimate. (d) Errors of rough estimate.

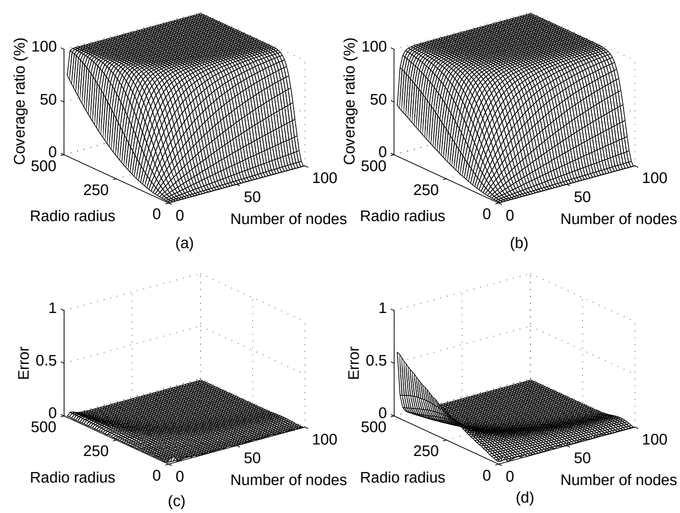

Fig. 4. Network coverage ratio in 1000×1000 rectangle, with the same ranges of n and r as with Fig. 2. (a) Results estimated by Eq. (2). (b) Results obtained from simulations (averaged over 10,000 experiments). (c) Errors with Theorem 3. (d) Errors with Eq. (2).

References

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