Stability and Fairness of AP Selection Games in IEEE 802.11 Access Networks

Li-Hsing Yen, Jia-Jun Li, and Che-Ming Lin
Department of Computer Science and Information Engineering
National University of Kaohsiung, Kaohsiung, Taiwan 811, R.O.C.
Email: lhyen@nuk.edu.tw

***Abstract*—Wireless stations (WSs) in an IEEE 802.11 access network compete with each other for collective bandwidth offered by access points (APs). The competition involves selecting an AP with the consideration of potential link rate and workload status. From the perspective of system, a good AP selection policy should be stable, increase overall system throughput, and maintain bandwidth fairness among WSs. This paper models AP selections under the framework of game theory, where each WS's sole goal is to maximize its achievable throughput. The achievable throughput depends on not only the number of WSs that associate with the same AP but also the set of link rates these WSs use: it is not a monotonically-decreasing function of WS population when considering the effect of performance anomaly. We have proven the stability of this game (Nash equilibrium), and shown that selfish behavior of individual WS in fact improves overall bandwidth fairness among WSs. Thorough simulations were conducted to demonstrate the validity of the analytical results and compare the performance of the proposed game with that of counterparts.**

I. INTRODUCTION

IEEE 802.11 wireless local area networks have been widely deployed as wireless infrastructures providing data access services in home, corporate, and public environments. In such environments, a wireless station (WS) with an IEEE 802.11 interface sends and receives frames via an access point (AP) to network infrastructure, and all APs in service constitute an access network. However, traffic load in an access network may not be fairly shared by all serving APs due to the uncoordinated nature of AP selections among WSs. More specifically, WSs typically select and associate with an AP with the highest received signal strength. This problem motivates many load-balancing schemes for IEEE 802.11 networks [1] with a design goal to make WS-AP associations load-aware, preventing WSs from making associations with congested APs. The ultimate goal is to either increase overall system throughput or maintain bandwidth fairness among WSs.

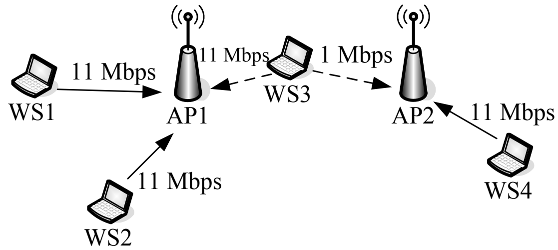
In this paper, we analyze the problem of AP selections under the framework of game theory. Game theory provides a mathematical modeling for the study of competition strategies in a game where players have conflicting benefits or goals. For the last decade, game theory has been used to analyze duty/resource sharing problems in wireless networks [2]. In these games, selfish players usually bring in undesired results (uneven load distribution or unfair resource share), and researchers have to introduce incentive or punitive mechanisms to force cooperation among players. For example, a commonly-adopted mechanism is to design a synthetic utility function for players that penalizes selfish behaviors. The goal is to let games naturally fall into stable states called Nash equilibria where system's interest could potentially benefit.

Our framework differs from previous ones in that WSs select and re-select APs merely for their own interest (specifically, achievable throughput that a WS may receive from a selection). No other external incentive/punitive mechanisms are introduced to ensure stability or fairness. The purpose of this research is to study the properties of the proposed AP selection game. We shall determine whether a Nash equilibrium exists even in this context, which eliminates the possibility of unstable association transitions (change of AP selections). Furthermore, we shall explore if the selfish yet rational behaviors of WSs under the proposed framework could improve bandwidth fairness. We shall also present simulation results for numerical analyses on the properties of the proposed game model and other alternatives.

The remainder of this paper is organized as follows. Background information and related work are presented in Section II. Section III analyzes several properties of the proposed game including stability and fairness. In Section IV, simulation results of the proposed game are discussed and compared with other alternatives. Section V concludes this paper.

II. BACKGROUND AND RELATED WORK

WSs in an IEEE 802.11 access network essentially compete for bandwidth offered by APs. Clearly, a WS's utility depends on not only its own association choice, but also other WS's. This is why game theory becomes a useful tool to apply here. Intuitively, WSs should select an AP that is the least crowded to maximize its achievable throughput. Games with player's objective defined to minimize the number of other users that share the same selection are known to be *crowding games* [3]. In the literature, crowding games has been used to model network selections by mobile users [4], [5]. However, this framework does not well apply to IEEE 802.11 networks as achievable throughput of WSs in an AP is not necessarily a monotonically-decreasing function of WS population there. The irregularity comes from two design features of IEEE



WS3's choice	Achievable throughput (Mbps)				Total (Mbps)
	WS1	WS2	WS3	WS4	
AP1	2.67 (2.21)	2.67 (2.23)	2.67 (2.23)	8.01 (6.57)	16.02 (13.24)
AP2	4.05 (3.34)	4.05 (3.38)	0.83 (0.75)	0.83 (0.84)	9.76 (8.31)

Fig. 1. A scenario illustrating performance anomaly. Achievable throughputs are based on the analysis of [6] while those in parentheses were obtained through simulation with ns2.

802.11. One is its non-deterministic MAC (Medium Access Control) scheme, which does not guarantee any bandwidth share to participants. The other is the provision of multiple link rates in IEEE 802.11 a/b/g networks, which may give rise to an undesirable phenomenon called *performance anomaly* [6]. Performance anomaly refers to the effect that when links operating at different rates coexist within an AP, throughputs of high-rate links will all degrade to the level of the lowest-rate link. Performance anomaly not only impairs achievable throughputs of WSs, but also makes AP's actual capacity a variable. Consider the example of Fig. 1, where two IEEE 802.11b APs are serving four WSs. WS3 there could choose either AP1 or AP2 to associate with. We can see that selecting AP1 yields a better result, though AP1 is more crowded than AP2. AP1 is also a better choice from the perspective of system's benefit, as selecting AP1 has a higher total achievable throughput than selecting the counterpart. Perception of performance anomaly can yield better performance result. But this cannot be characterized in crowding games.

The AP selection problem under consideration is modeled as a *noncooperative dynamic* game. In a noncooperative game, players do not cooperate with each other to seek system's benefit. A noncooperative game is dynamic if players take turns to make their decisions, knowing what decisions have already been done. In our model, an associated WS will re-associate with another AP if that re-association improves its achievable throughput. The achievable throughput in recognition of the effect of performance anomaly can be computed with analytical results from prior work in [6], [7]. Here we assume WSs pursuing its own throughput improvement rather than the balance of workloads among APs (e.g., [8]). Although a lightly-loaded AP in principle offers a high achievable throughput and selecting an AP with the least load helps load balancing among APs, we argue that, from WS's perspective, AP selections based on achievable throughput are more straightforward and "natural" than AP selections based on load balancing. Several other approaches also proposed AP selections based on achievable throughput (potential bandwidth) [9], [10], [11]. Another issue of load-based AP selections comes from the fact that the notion of AP's load is not well defined in IEEE 802.11 networks. It could be the number of WSs associating with an AP, frame drop rate of AP's transmission queue during real-time sessions [12], or the total time that an AP takes to provide each WS one unit of traffic [13], [8].

Although there have been many approaches proposed for AP selections, only few of them treat the problem under the framework of game theory. Mittal *et al.* [14] introduced an AP selection game which differs from our setting in that WSs may need to travel some distance to reach an AP. The cost of an AP selection is measured by the AP's load and the traveling distance required by that selection. With this cost model, Mittal *et al.* proposed a simple greedy algorithm that brings the game to a Nash equilibrium under the condition of even WS distribution and absence of dynamic WS arrivals and departures. However, the ability to measure physical distance between WSs and APs, as required by this model, is not yet a primitive feature in today's wireless networks. Shakkottai *et al.* [15] studied the problem of a WS associating with multiple APs and splitting its traffic among these APs (*link-layer multihoming*). They used the model of *population game* [16], which implies that the impact of individual WS's selection on other WS's utilities is infinitesimal. Although link-layer multihoming is possible for WSs using a single wireless interface card [17], this technique is not yet mature and widely adopted. The population game model also does not generally apply to IEEE 802.11 networks. Jiang *et al.* [18] considered base station (BS) selections by mobile users, where each user selfishly chooses a BS that gives her the highest achievable throughput. This work assumes that the throughput each user can receive is controlled by the BS, and that the number of users is enormous so as to apply the population game model. The ability to control user's throughput share by the BS is untenable in native IEEE 802.11 networks. The assumption of numerous users may not hold.

Besides throughput, *fairness* is also a typical criterion for AP selection problems. In the context of bandwidth sharing, *max-min* [19] is a commonly-adopted metric for fairness particularly when bandwidth requestors have different bandwidth demands. With an objective to maximize the minimum share of a requestor whose demand is not fully satisfied, basic principles of max-min fairness are to allocate bandwidth to requestors in increasing demands, to ensure no requestor receives bandwidth more than its demand, and to equally split the remaining bandwidth to requestors with unsatisfied demands. If we use a tuple to denote the set of allocated bandwidth of every requestor sorted in a nondecreasing order, then a bandwidth allocation is max-min fair when the corresponding tuple has the highest lexicographical value[1] among all.

A similar notion, *min-max* fairness, can be defined for the sharing of workloads among APs. A distribution of workloads

[1]For any two n-tuples of numbers $T = (t_1, t_2, \cdots, t_n)$ and $T' = (t'_1, t'_2, \cdots, t'_n)$, T has a higher lexicographical value than T' if $\exists k \in \{1..n\} : t_k > t'_k$ and, if $k > 1$, $\forall i : 1 \leq i < k :: t_i = t'_i$.

is min-max fair if the tuple denoting the set of workloads of every AP sorted in a non-increasing order has the lowest lexicographical value among all possibilities. Bejerano *et al.* [13] have studied AP selections that achieve min-max fairness of AP workloads. They proved that, unless link-layer multi-homing is allowed, a min-max load balanced association does not imply a max-min fair bandwidth allocation and vice versa.

Max-min fairness well applies to cases where resource requestors have limited demands. In our problem setting, however, every WS has an unlimited bandwidth demand; it could actually consume all bandwidth available to it. For this kind of bandwidth sharing, *balance index* [20] can be used to quantify the fairness of bandwidth share among all competitors. For a bandwidth allocation consisting of n portions numbered 1 to n, let B_i, $1 \leq i \leq n$, denote the amount of bandwidth allocated to the ith portion. The balance index β is defined as

$$\beta = \frac{(\sum B_i)^2}{n \times \sum B_i^2}. \quad (1)$$

The value of β becomes 1 when all requestors get an equal share, and it approaches $1/n$ in case of extremely unbalanced allocations. Balance index can be related to max-min fairness in the sense that, when WSs all have unlimited bandwidth demands, a bandwidth allocation with $\beta = 1$ is also max-min fair. However, the converse does not hold generally.

III. AP Selection Game

We consider an IEEE 802.11 network consisting of n WSs and m APs. Neighboring APs are assumed to operate at different (non-overlapping) frequency channels such that there is no interference among APs. Let $A = (a_1, a_2, \cdots, a_m)$ and $W = (w_1, w_2, \cdots, w_n)$ be the tuples of all APs and WSs, respectively. We assume that each WS can access at least one AP and denote the set of APs that w_i can access (i.e., the strategy set of w_i) by A_i, where $1 \leq i \leq n$. For a possible AP-WS association, the WS's *utility* is defined to be the achievable throughput of the WS resulted from that association.

We define a *configuration* (a strategy profile) to be an n-tuple $C = (c_1, c_2, \cdots, c_n)$, where $c_i \in A_i$ represents w_i's association choice. For a specific w_i, we may sometimes express C as $C = (c_i, C_{-i})$, where $C_{-i} = (c_1, c_2, \cdots, c_{i-1}, c_{i+1}, \cdots, c_n)$ denotes all other WS's associations other than w_i's. Function $u_i(C)$ gives w_i's utility with respect to configuration C. We shall explore how to evaluate $u_i(C)$ in the next subsection. The AP selection game $\Gamma = [W; A; \{u_i\}_{i=1}^n]$ can be formally defined by $\max_{c_i \in A_i} u_i(c_i, C_{-i})$ for all $i = 1, 2, \cdots, n$. For the ensuing discussion, most of the symbols used are summarized in Table I.

A WS may seek an optimal association decision if the decision is made by considering all possible decisions that other WSs could make and what rules other WSs would follow to make their decisions. An equivalent approach is to examine all possible configurations to find the best strategy for a particular WS. The proposed game takes a simpler model instead, where WSs do not predict or analyze other WS's intentions or best strategies: they only *respond* to other WS's actions. With the knowledge of all other WS's choices, a WS conducts an association change only if that change maximizes the net increase of its utility among all possible re-association choices under the assumption that all other WSs stay unchanged. Formally, the *best response function* for WS w_i is $b_i(C_{-i}) = \{c_i \in A_i | \forall c'_i \in A_i : u_i(c_i, C_{-i}) \geq u_i(c'_i, C_{-i})\}$. This re-association rule may be shortsighted in the sense that the increase of utility is evaluated without considering possible responses from other WSs. Theoretically, the decision of association change may turn out to be a loss when later other WSs respond with their association changes. Fortunately, this is not a one-shot game; the original WS may recover its loss by making another re-association. Our major concern is whether such interactions result in nonstop chain reactions, and how the game evolves in terms of bandwidth fairness and overall system throughput.

TABLE I
Partial list of notations

Notation	Meaning
m	Number of APs
n	Number of WSs
A	The tuple of all APs; $A = (a_1, a_2, \cdots, a_m)$
W	The tuple of all WSs; $W = (w_1, w_2, \cdots, w_n)$
A_i	The set of APs that w_i can associate with
W_j	The set of WSs that associate with a_j
c_i, c_i^*, c'_i	The AP that w_i associates with
$u_i(C)$	w_i's utility with respect to configuration C
Σ	The configuration space (set of all configurations)
$t(a, C)$	The throughput of any WS residing in AP a with respect to configuration C.

In our game model, a transition from one configuration to another occurs when some WS conducts an association change. For simplicity, we assume that only one association change is conducted at a time; simultaneous transitions are serialized in some arbitrary order. Denote the transition relation by '$\rightsquigarrow$'. Formally, for any two configurations C_i and C_j, $C_i \rightsquigarrow C_j$ if $u_r(C_i) < u_r(C_j)$, where w_r is the only WS that has different association choices between C_i and C_j.

A. Utility: Achievable Throughput

For each WS w_i in the proposed game, its utility function $u_i(C)$ is defined to be the achievable throughput of w_i in configuration C. Heusse *et al.* [6] have analyzed achievable throughputs of WSs under IEEE 802.11 multi-rate environment. Their analysis assumes no interference among neighboring APs and can be summarized as follows. For any WS w_i operating at link rate r_i, its MAC-layer throughput can be expressed as

$$X_i = U_i \times \frac{s_d}{r_i T_i} \times r_i, \quad (2)$$

where U_i is the fraction of time w_i is able to access the medium, T_i is overall transmission time (counting protocol overhead, transmission time, and the time spent in contention procedure) for a single frame sent by w_i, and s_d is the size (in bits) of the frame. By definition, $U_i = T_i / I_i$, where I_i is the average time between two consecutive transmissions of w_i. Therefore, (2) can be simplified as

$$X_i = \frac{s_d}{I_i}. \quad (3)$$

Let W_j denote the set of all WSs that associate with AP a_j. One property of 802.11 MAC scheme is that all WSs in W_j have equal long-term channel access probability regardless of their link rates. In case of saturated traffic (i.e., every WS always has packets to transmit), this means that each $w_i \in W_j$ is expected to have an I_i value that comprises T_k for all $w_k \in W_j$ and the expected time spent in all possible collisions among WSs in W_j during I_i. Formally,

$$\forall w_i \in W_j : I_i = \left(\sum_{w_k \in W_j} T_k \right) + \delta(W_j), \tag{4}$$

where $\delta(W_j)$ is the expected time spent in all possible collisions among WSs in W_j during I_i. T_k consists of a rate-independent part (corresponding to protocol overhead and the time spent in contention procedure) and a variable-length part (transmission time) that depends on frame length s_d and link rate r_k. The duration of a collision is also dominated by the lowest rate of WSs involved in the collision. A WS obtains its maximal throughput when it always has packets to transmit and each frame is of the maximal frame size. Therefore, all WSs that associate with the same AP receive equal amount of achievable throughput that is determined by the mixture of their link rates but dominated by low-rate links. Consequently, the performance of high-rate links is effectively dragged down by low-rate links. The analysis presented in [7] shows similar conclusions.

Let $t(a, C)$ be the achievable throughput of any WS residing in AP a with respect to configuration C. By (3) and (4), we know that

$$t(a, C) = \frac{s_d}{\left(\sum_{c_k = a} T_k\right) + \delta(\{c_k = a\})}. \tag{5}$$

In the proposed AP selection game, $u_i(c_i, C_{-i}) = t(c_i, C)$ and the game can thus be defined by

$$\max_{c_i \in A_i} u_i(c_i, C_{-i}) = \max_{c_i \in A_i} \frac{s_d}{\left(\sum_{c_k = c_i} T_k\right) + \delta(\{c_k = c_i\})} \tag{6}$$

for all $i = 1, 2, \cdots, n$.

Without loss of generality, assume that $C_i \rightsquigarrow C_j$ because some WS w_r changes its AP from a_k to a_l. Since $C_i \rightsquigarrow C_j$ implies $u_r(C_i) < u_r(C_j)$, we have

$$t(a_k, C_i) < t(a_l, C_j). \tag{7}$$

Note that all and only all WSs associating with either a_k or a_l have their achievable throughputs changed by $C_i \rightsquigarrow C_j$. Therefore, if we are concerned with total achievable throughput in the system, the net increase due to the transition is

$$\begin{aligned}(d-1) \times t(a_k, C_j) - d \times t(a_k, C_i) + \\ e \times t(a_l, C_j) - (e-1) \times t(a_l, C_i),\end{aligned} \tag{8}$$

where d is the number of WSs associating with a_k in C_i and e is the number of WSs (including w_r) associating with a_l in C_j. Clearly, Eq. (7) does not guarantee a positive net increase. In fact, an association change in the proposed AP selection game may lead to degradation of total achievable throughput in the system.

For comparison purpose, we also define a *public-interest first* (PIF) re-association model, where a WS makes an association change only if that association results in an increase of total achievable throughput in the system. In case of multiple candidates, the WS chooses the one that results in the maximal net increase. Although WS's own benefit may be sacrificed in this model, the system is always benefited from re-associations.

B. Stability

Definition 1: Nash Equilibrium: Given a game $\Gamma = [W; A; \{u_i\}_{i=1}^n]$, a configuration $C^* = (c_1^*, c_2^*, \cdots, c_n^*)$ is a Nash equilibrium if $\forall i \in \{1..n\} : \forall c_i \in A_i :: u_i(c_i^*, C_{-i}^*) \geq u_i(c_i, C_{-i}^*)$.

In other words, Nash equilibrium is a configuration where no WS can further increase its own utility by unilaterally changing its choice. Nash equilibrium is not necessarily a Pareto optimal strategy. A configuration $C = (c_1, c_2, \cdots, c_n)$ is Pareto optimal if and only if there exists no other configuration $C' = (c_1', c_2', \cdots, c_n')$ such that $\forall i \in \{1..n\} : u_i(C') \geq u_i(C)$ and $\exists j \in \{1..n\} : u_j(C') > u_j(C)$.

Recall that in our model, an associated WS can re-associate with another AP if that re-association improves its achievable throughput. The re-association action may trigger another WS's re-association and so on. If Nash equilibria do not exist in this game, re-association activities will last and the system cannot enter a stable state. By contrast, stability in the PIF re-association model is always guaranteed as it is impossible to unlimitedly increase total achievable throughput of the system. We shall now show the existence of Nash equilibria in the proposed AP selection game.

Let $\Sigma = A_1 \times A_2 \times \cdots \times A_n$ be the configuration space, i.e., the set of all possible configurations. If there exists no Nash equilibrium, then for any configuration $C_i \in \Sigma$, there must exist another configuration $C_j \in \Sigma$ such that $C_i \rightsquigarrow C_j$. Since the strategy space is finite, nonexistence of Nash equilibrium implies that there must be a series of configurations $C_1', C_2', \cdots, C_p'$, where $p \leq k$, such that $C_1' \rightsquigarrow C_2'$, $C_2' \rightsquigarrow C_3'$, $\cdots, C_p' \rightsquigarrow C_1'$. We shall prove the existence of Nash equilibrium by showing that such series does not exist.

For each configuration $C_i \in \Sigma$, let $T(C_i) = (\alpha_i^1, \alpha_i^2, \cdots, \alpha_i^m)$ be an m-tuple of APs, where $\{\alpha_i^1, \alpha_i^2, \cdots, \alpha_i^m\}$ is an ordered set of all APs such that $t(\alpha_i^1, C_i) \leq t(\alpha_i^2, C_i) \leq \cdots \leq t(\alpha_i^m, C_i)$[2]. Let $\Theta = \{T(C) | C \in \Sigma)\}$. We also define a binary relation $\prec$ on Θ as follows. For $T(C_i), T(C_j) \in \Theta$, we have $T(C_i) \prec T(C_j)$ if $\exists k \in \{1..m\} : t(\alpha_i^k, C_i) < t(\alpha_j^k, C_j)$ and, if $k > 1$, $\forall l : 1 \leq l < k :: t(\alpha_i^l, C_i) = t(\alpha_j^l, C_j)$. It is not hard to see that '$\prec$' is a precedence relation [21], i.e., it is antisymmetric and transitive.

Theorem 1: $\forall C_i, C_j \in \Sigma : C_i \rightsquigarrow C_j \Rightarrow T(C_i) \prec T(C_j)$.

Proof: Without loss of generality, assume that $C_i \rightsquigarrow C_j$ because some WS w_r changes its AP from a_k to a_l. Let a_k

[2]For any AP α_i^j that draws no WS in C_i, we define $t(\alpha_i^j, C_i)$ to be the nominal capacity of one AP (11 Mbps in case of IEEE 802.11b) so that such APs are always ranked after any AP with one or more WS associations.

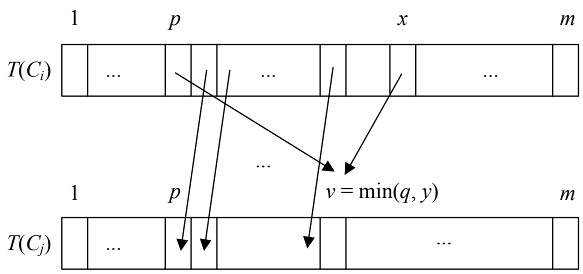


Fig. 2. Rank mapping from $T(C_i)$ to $T(C_j)$ when $v = \min\{q, y\} > p$

be the pth and qth element in $T(C_i)$ and $T(C_j)$, respectively. In other words, $a_k = \alpha_i^p = \alpha_j^q$. Similarly, let $a_l = \alpha_i^x = \alpha_j^y$. $C_i \rightsquigarrow C_j$ implies that $u_r(C_i) < u_r(C_j)$, which in turn implies that

$$t(\alpha_i^p, C_i) < t(\alpha_j^y, C_j). \tag{9}$$

If there exists at least one WS associating with a_k after the association migration of w_r, then $t(a_k)$ will be increased, meaning that

$$t(\alpha_i^p, C_i) < t(\alpha_j^q, C_j). \tag{10}$$

If there is no other WS associating with a_k after w_r's leave, then $t(\alpha_j^q, C_j)$ equals to the nominal capacity of one AP as defined, and (10) still holds. By (9), (10), and the fact that a_k and a_l are the only two APs whose throughput is changed by $C_i \rightsquigarrow C_j$, the first $p-1$ elements in $T(C_i)$ hold their ranks in $T(C_j)$. Thus we have

$$\forall s : 1 \leq s \leq p-1 :: t(\alpha_i^s, C_i) = t(\alpha_j^s, C_j). \tag{11}$$

Now consider the relation between $v = \min\{q, y\}$ and p. By (9), (10), and (11), it is impossible that $v < p$. If $v = p$, then we have the proof by (11) and either (9) or (10). If $v > p$, then α_i^p must change its rank from the pth element in $T(C_i)$ to at least the vth element in $T(C_j)$, and all APs in between change their ranks accordingly (Fig. 2). That is,

$$\forall s : p \leq s \leq v-1 :: t(\alpha_j^s, C_j) = t(\alpha_i^{s+1}, C_i). \tag{12}$$

Since $\forall s : p \leq s \leq v-1 :: t(\alpha_i^{s+1}, C_i) \geq t(\alpha_i^s, C_i)$, (12) implies that

$$\forall s : p \leq s \leq v-1 :: t(\alpha_i^s, C_i) \leq t(\alpha_j^s, C_j). \tag{13}$$

Moreover, from (12) we know that $t(\alpha_j^{v-1}, C_j) = t(\alpha_i^v, C_i)$. This together with the fact $t(\alpha_j^{v-1}, C_j) \leq t(\alpha_j^v, C_j)$ implies that

$$t(\alpha_i^v, C_i) \leq t(\alpha_j^v, C_j). \tag{14}$$

Equations (13) and (14) can be merged into

$$\forall s : p \leq s \leq v :: t(\alpha_i^s, C_i) \leq t(\alpha_j^s, C_j), \tag{15}$$

which implies that either

$$\forall s : p \leq s \leq v :: t(\alpha_i^s, C_i) = t(\alpha_j^s, C_j) \tag{16}$$

or

$$\exists s : p \leq s \leq v :: t(\alpha_i^s, C_i) < t(\alpha_j^s, C_j) \tag{17}$$

holds. If (16) holds, then we have $t(\alpha_i^s, C_i) = t(\alpha_i^{s+1}, C_i)$ for all s, $p \leq s \leq v-1$, by (12), which in turn implies that $t(\alpha_i^p, C_i) = t(\alpha_j^v, C_j)$. The derived result contradicts with either (9) (when $v = y$) or (10) (when $v = q$). Therefore, only (17) holds. The theorem is thus proven by (11), (15), and (17). ■

Half of the proof deals with the case that different APs may provide identical throughputs for WSs associating with them. If this case were not considered, the equality in (13) would not hold and all the subsequent arguments would not be needed.

If there is any series of configuration transitions $C'_1, C'_2, \cdots, C'_p$, where $p \leq k$, in the proposed game such that $C'_1 \rightsquigarrow C'_2$, $C'_2 \rightsquigarrow C'_3$, $\cdots, C'_p \rightsquigarrow C'_1$, then by Theorem 1 we have $T(C'_1) \prec T(C'_2)$, $T(C'_2) \prec T(C'_3), \cdots, T(C'_p) \prec T(C'_1)$. It follows that $T(C'_2) \prec T(C'_1)$ as $\prec$ is transitive, which leads to a contradiction since $\prec$ is also antisymmetric. Therefore, Theorem 1 implies that any loop of configuration transitions is impossible, and suffices to be a proof for the existence of Nash equilibria in the proposed game. More specifically, starting from any configuration, re-association activities made by WSs always end up with a configuration where no WS can further increase its own utility by unilaterally changing its choice.

Note that different initial configurations may end up with different Nash equilibria. Even with the same initial configuration, different re-association orders may lead to different stable configurations. Fig. 3 gives an example, where both WS1 and WS2 have the motivation to re-associate with AP2. Depending on which one moves first, two possible Nash equilibria can be reached.

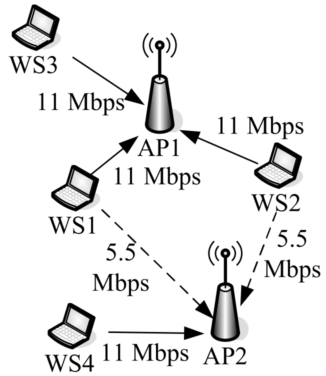
C. Fairness

We shall now address the fairness issue of the game. The definition of max-min fairness refers to only one configuration. To quantify the relative degree of fairness for *every* feasible configuration, we propose measuring the lexicographical value of the corresponding utility tuple. More precisely, each configuration $C_i \in \Sigma$ corresponds to a *utility tuple* $U_i = (\mu_i^1, \mu_i^2, \cdots, \mu_i^n)$ which is obtained by sorting $\{u_j(C_i)\}_{j=1}^n$ in a nondecreasing order.

Definition 2: Given two configurations C_i and C_j with respective utility tuples $U_i = (\mu_i^1, \mu_i^2, \cdots, \mu_i^n)$ and $U_j = (\mu_j^1, \mu_j^2, \cdots, \mu_j^n)$, we say that C_i is *lexicographically fairer* than C_j if U_i has a higher lexicographical value than U_j, i.e., $\exists k \in \{1..n\} : \mu_i^k > \mu_j^k$ and, if $k > 1$, $\forall l : 1 \leq l < k :: \mu_i^l = \mu_j^l$.

Intuitively, C_i is lexicographically fairer than C_j if the lowest utility in C_i is larger than that in C_j, or the lowest utility in C_i is equal to that in C_j but the second lowest utility in C_i is larger than that in C_j, and so on. This definition is consistent with max-min fairness in the sense that a configuration is max-min fair if and only if it is lexicographically fairer than any others.

We can derive U_i from $T(C_i)$ by seeing that all WSs associating with the same AP receive equal throughput. Let $w(\alpha_i^k)$ be the number of WSs associating with AP $\alpha_i^k \in T(C_i)$, where $1 \leq k \leq m$. Given $T(C_i)$, we let each AP α_i^k map to $w(\alpha_i^k)$ consecutive elements in U_i. Specifically, the following function returns the position of the first element in U_i that



Configuration	Achievable throughput (Mbps)				
	WS1	WS2	WS3	WS4	
Initial	2.67	2.67	2.67	8.01	
WS1 chooses AP2 first	2.93	4.05	4.05	2.93	(Nash Equilibrium)
WS2 chooses AP2 first	4.05	2.93	4.05	2.93	(Nash Equilibrium)

Fig. 3. An example illustrating the effect of re-association order. Achievable throughputs are based on the analysis of [6].

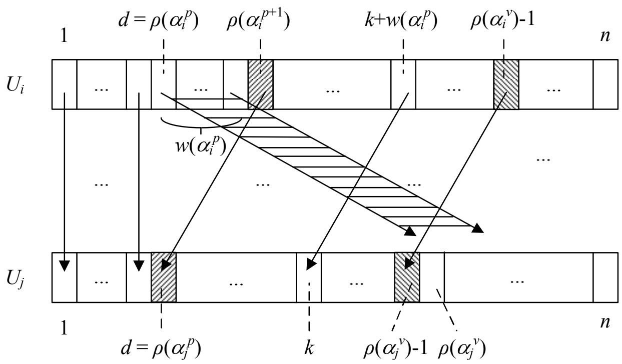


Fig. 4. Rank mapping from U_i to U_j when $v = \min\{q, y\} > p$

corresponds to a WS associating with α_i^k (if there is any WS associating with α_i^k):

$$\rho(\alpha_i^k) = \begin{cases} 1 & k = 1, \\ 1 + \sum_{l=1}^{k-1} w(\alpha_i^l) & 2 \leq k \leq m. \end{cases} \tag{18}$$

If $w(\alpha_i^k) = 0$, α_i^k maps to no element in U_i. Otherwise, all elements in U_i with ordinal numbers ranging from $\rho(\alpha_i^k)$ to $\rho(\alpha_i^{k+1}) - 1$, $1 \leq k \leq m-1$, have identical values $t(\alpha_i^k, C_i)$.

With the way to derive U_i from a given $T(C_i)$, we shall further prove that if $C_i \rightsquigarrow C_j$, then U_j also has a higher lexicographical value than U_i, meaning that C_j is lexicographically fairer than C_i. Consequently, configuration transitions in the proposed AP selection game always improve utility fairness.

Theorem 2: $\forall C_i, C_j \in \Sigma : C_i \rightsquigarrow C_j \Rightarrow U_i \prec U_j$.

Proof: U_i and U_j can be derived from $T(C_i)$ and $T(C_j)$, respectively, as stated above. We assume the same definitions of p, q, and y as in the proof of Theorem 1. Since the first $p-1$ APs in $T(C_i)$ hold their ranks in $T(C_j)$, all the first $\rho(\alpha_i^p) - 1$ elements in U_i are identical to the corresponding elements in U_j. That is,

$$\forall k : 1 \leq k \leq \rho(\alpha_i^p) - 1 : \mu_j^k = \mu_i^k. \tag{19}$$

Now consider $v = \min\{q, y\}$. If $v = p$, then the $\rho(\alpha_i^p)$th element in U_i is smaller than the corresponding element in U_j by either (9) or (10), and the proof is done. If $v > p$, then the position of either α_j^q or α_j^y is at least the vth in $T(C_j)$, which means all the $w(\alpha_i^p)$ elements associating with α_i^p are placed at least the $\rho(\alpha_j^v)$th position in U_j. It follows that

$$\forall k : p \leq k \leq v-1 :: \rho(\alpha_j^k) = \rho(\alpha_i^{k+1}) - w(\alpha_i^p), \tag{20}$$

$$\forall k : p \leq k \leq v-1 :: w(\alpha_j^k) = w(\alpha_i^{k+1}), \tag{21}$$

and

$$\forall k : \rho(\alpha_j^p) \leq k \leq \rho(\alpha_j^v) - 1 :: \mu_j^k = \mu_i^{k+w(\alpha_i^p)} \geq \mu_i^k. \tag{22}$$

See Fig. 4 for the rank mapping from U_i to U_j indicated by (20). Eq. (22) implies that either

$$\exists s : \rho(\alpha_j^p) \leq s \leq \rho(\alpha_j^v) - 1 :: \mu_j^s > \mu_i^s \tag{23}$$

or

$$\forall k : \rho(\alpha_j^p) \leq k \leq \rho(\alpha_j^v) - 1 :: \mu_j^k = \mu_i^{k+w(\alpha_i^p)} = \mu_i^k. \tag{24}$$

If (23) holds, the theorem is proven by (19), (22), and (23). If (24) holds, by (20) we have

$$\begin{aligned} & \forall k : \rho(\alpha_i^p) \leq k \leq \rho(\alpha_i^{v+1}) - w(\alpha_i^p) - 1 :: \mu_i^k = \mu_i^{k+w(\alpha_i^p)} \\ \Rightarrow \quad & \mu_i^{\rho(\alpha_i^p)} = \mu_i^{\rho(\alpha_i^p)+1} = \cdots = \mu_i^{\rho(\alpha_i^{v+1})-1}. \end{aligned} \tag{25}$$

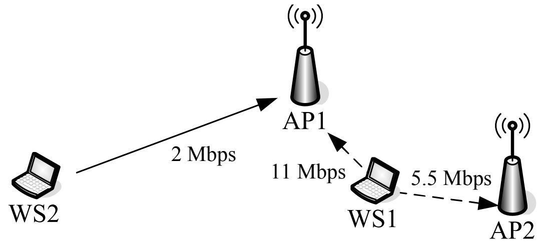
Let $d = \rho(\alpha_i^p)$ and $e = \rho(\alpha_j^v)$. The derivation of (26) is based on (18), (20), and (21):

$$\begin{aligned} & e = \rho(\alpha_j^{v-1}) + w(\alpha_j^{v-1}) \\ \Rightarrow \quad & e = \rho(\alpha_i^v) - w(\alpha_i^p) + w(\alpha_j^{v-1}) \\ \Rightarrow \quad & e = \rho(\alpha_i^v) - w(\alpha_i^p) + w(\alpha_i^v) \\ \Rightarrow \quad & e = \rho(\alpha_i^{v+1}) - w(\alpha_i^p) \leq \rho(\alpha_i^{v+1}) - 1. \end{aligned} \tag{26}$$

By (25) and (26) we have $\mu_i^e = \mu_i^d$. Furthermore, $\mu_j^e > \mu_i^d$ by (9) and (10). Therefore, we have $\mu_j^e > \mu_i^e$. The theorem is thus proven. ■

This proof is similar to that of Theorem 1 in the sense that a lot of efforts are devoted to deal with the case of identical utility values among adjacent elements in a utility tuple. The key point is, despite the existence of a continuous series of identical utility values, we can always find a break point in the corresponding utility tuple that exhibits a difference between U_i and U_j.

It should be noted that the definition of lexicographical fairness does not always comply with the definition of balance index. Refer to the scenario shown in Fig. 5, where WS1 could select either AP1 or AP2 to associate with. The former



WS1's choice	Achievable throughput (Mbps)		Utility tuple	Balance index
	WS1	WS2		
AP1	1.48	1.48	$\langle 1.48, 1.48 \rangle$	1
AP2	4.61	1.85	$\langle 1.85, 4.61 \rangle$	0.846

Fig. 5. A scenario illustrating the difference between lexicographical fairness and balance index

TABLE II
CONVERSION OF DISTANCE TO LINK RATE

Range of distance d (m)	Link rate (Mbps)
$0 \leq d < 50$	11
$50 \leq d < 80$	5.5
$80 \leq d < 120$	2
$120 \leq d < 150$	1
$d \geq 150$	0

selection results in a higher balance index than the latter while the latter is lexicographically fairer than the former. Despite the existence of such counterexample in a synthesized setting, in the next section we shall show through simulations that configuration transitions in the proposed AP selection game generally improve bandwidth fairness in terms of balance index.

IV. NUMERICAL RESULTS

We conducted extended simulations to study the properties of the proposed game. The simulation setting is as follows. APs form a square grid in a 600×600 (m^2) area with the dimension of the sides of the grid squares set to 2 to 15. Neighboring APs (also a border AP and the border of the area) are separated with equal distance. WSs are randomly uniformly distributed over the same region with the number of WSs varied 50 to 500 in increments of 50. The link rate between a WS and an AP is based on IEEE 802.11b and determined by their in-between distance (Table II). We also preclude unconnected WSs by randomly relocating such WSs. For each setting, 1000 trials were made for an average result.

We let each WS select an AP based on received signal strength (RSS) initially. Here all APs are assumed identical transmitting power, and a simple path-loss model is adopted where RSS decreases with the square of the traveling distance of the signal. After its initial association, a WS selectees an AP to re-associate with following either the proposed game model or the PIF model. When multiple WSs are eligible to make an association change, we randomly select one WS at

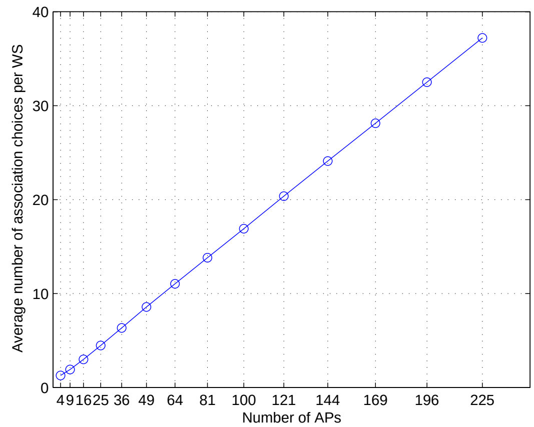


Fig. 7. Average number of association choices per WS

a time to do so. The achievable throughput of each WS is calculated based on the analysis of Heusse *et al.* [6].

A. Number of Re-associations

Figure 6 shows the total number of re-associations in the proposed game and in the PIF model for each possible setting. The re-association activities in the proposed game all stop after a limited number of times, which validates our analysis. In general, WSs in the PIF model experience more re-associations than in the proposed game model. The average number of re-associations per WS in the proposed game is 0.367 with standard deviation 0.131. For the PIF model, the average value and standard deviation are 0.393 and 0.185, respectively.

Three key factors govern the number of re-associations: the nature of the re-association policy, the number of association choices, and the degree of association competition. The first factor is model dependent and explains the difference between Fig. 6(a) and Fig. 6(b). The number of re-associations, as a result of competitions, generally increases with the number of association choices and the degree of association competition. Given a certain experiment setting, the expected number of association choices owned by a WS can be measured by the average number of APs accessible to a WS. It is irrelevant to the total number of WSs deployed. Fig. 7 displays how the average number of association choices changes with the number of APs. The degree of competition counts the number of competitors each WS is expected to face for a particular WS-AP association. For WS w_i to associate with AP a_j, the degree of competition depends on not only the number of other WSs that can also associate with a_j, but also the likelihood that these potential competitors actually do it. Without resort to the knowledge of a specific re-association model, let us assume that every AP in A_i will be chosen by w_i with equal preference and this holds for every WS w_i. It follows that the expected number of competitors w_i has to face is

$$\sum_{a_j \in A_i} \left(\frac{1}{|A_i|} \sum_{w_k \in P_j, k \neq i} \frac{1}{|A_k|} \right), \tag{27}$$

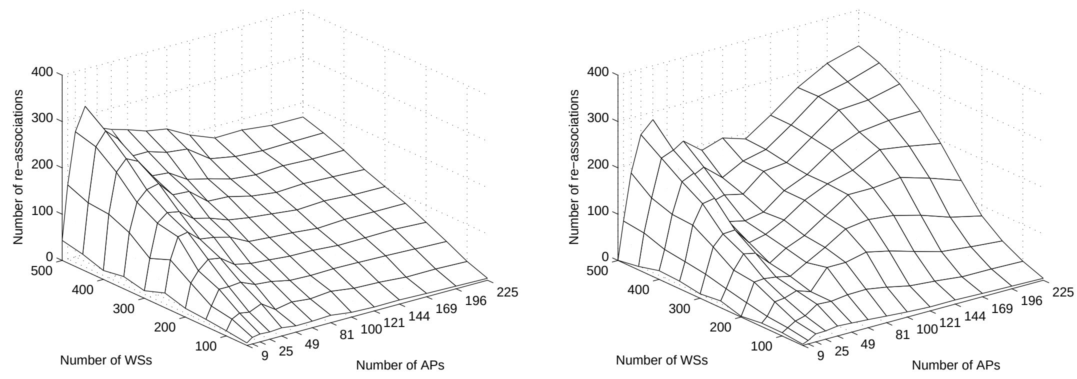


Fig. 6. (a) Number of re-associations in the proposed game before Nash equilibrium. (b) Number of re-associations in the PIF model.

where $P_j = \{w_k | a_j \in A_k\}$ is the set of WSs that can associate with a_j. We take the average value of (27) over all WSs as the the expected degree of competition. Fig. 8(a) shows the measured results. Clearly, the average degree of competition increases with the number of WSs but decreases with the number of APs. Since the average degree of competition has a linear relationship with the number of WSs, we divide the former by the latter and get the result of Fig. 8(b). In the following, we explain the results of Fig. 6 with the help of Figs. 7 and 8.

- For a fixed number of WSs, the expected number of choices is in proportion to the number of APs (Fig. 7). However, in Fig. 6 we observe a rather high re-association count when only 4 to 25 or 36 APs are deployed. This must be contributed by the extremely-high degree of competition in that range (Fig. 8(b)). When the number of APs is further increased, the total re-association count does not rise further but rather declines slightly. This can be justified as the extremely-low degree of competition cancels out the trend of increasing re-associations due to the increase of the expected number of choices. Consequently, WSs experience even fewer re-associations to reach Nash equilibria.
- For a fixed number of APs, the expected number of choices is fixed while the expected degree of competition is proportional to the population of WSs (Fig. 8(a)). When the number of WSs is small, modest competitions and few re-associations are observed. As more WSs are involved, competitions among WSs become intense, giving rise to more interactive re-associations. Consequently, total re-association count roughly increases with the number of WSs. The increasing rate with the PIF model, however, is generally higher than that with our game model, as Fig. 9 indicates.

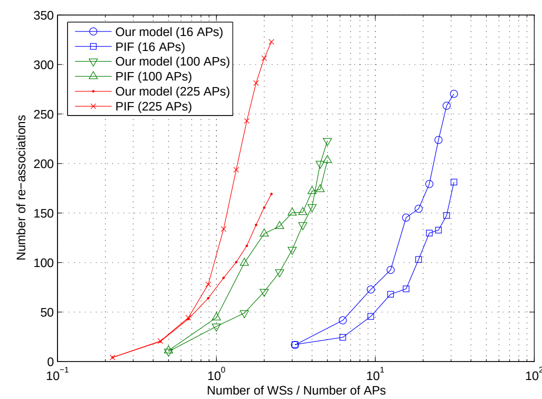
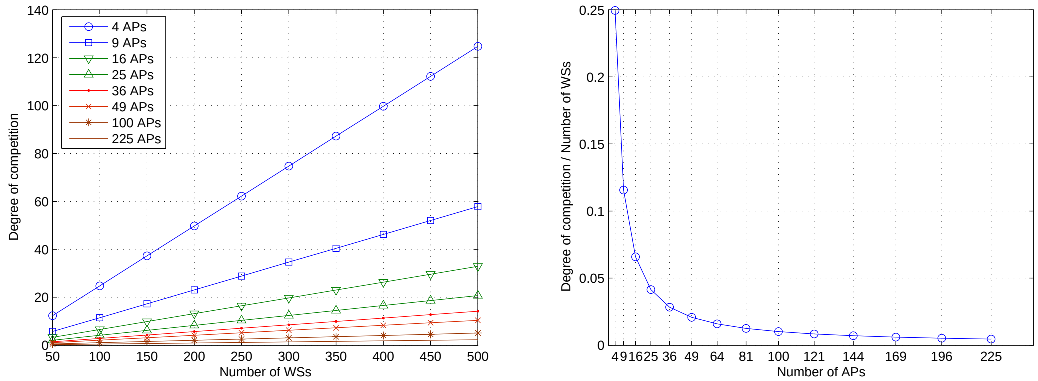


Fig. 9. Comparison of re-association increase rate between the proposed game and the PIF models.

B. Balance Index

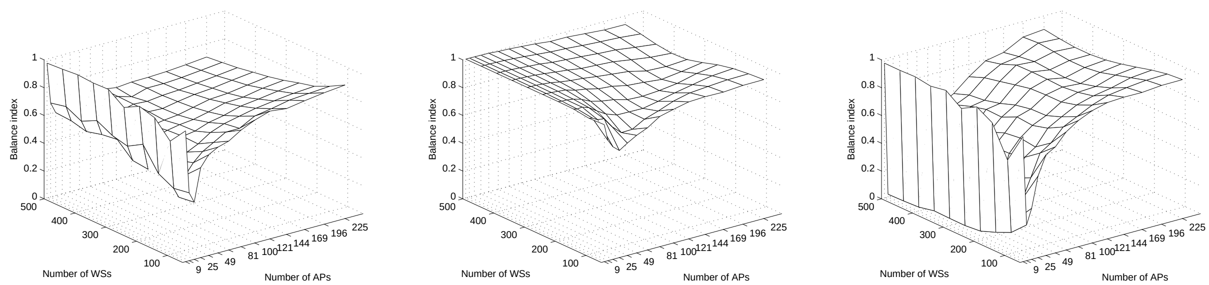
We also measured balance indices for both AP selection models. For each trial, the balance index was measured after the initial RSS-based associations and also after the stop of all re-association activities. Fig. 10(a) displays the balance indices measured after the initial RSS-based associations. We can see that when only four APs are deployed, the RSS-based association policy results in rather high balance indices. This can be understood as when only few APs are accessible to numbers of WSs, all APs are overpopulated and offer a similar amount of bandwidth share (which is extreme low) to each WS. When more APs are deployed, the workloads of APs become diverse due to the fact that RSS-based association policy is not load aware. Different WSs therefore receive different amounts of achievable throughput, which explains the sharp drop of the balance index when the number of APs is increased from 4 to 16. When the number of APs is further increased or when the number of WSs is decreased, the expected degree of competition and thus the impact of performance anomaly both lessen. It turns out that the difference of achievable throughputs among WSs diminishes. This justifies both the rise of balance indices with the number of APs and the descent of balance indices with the number of WSs in the right half of Fig. 10(a).

Figure 10(b) shows balance indices measured after the stop of the proposed game model while (c) shows the same results for the PIF model. Observe that the game model raises the balance indices for all experimental settings but the PIF model



(a) (b)

Fig. 8. (a) Expected degree of competition. (b) Expected degree of competition divided by the number of WSs vs. the number of APs.



(a) (b) (c)

Fig. 10. Balance index (a) after the initial RSS-based associations and after the stop of re-association activities in (b) the proposed game and (c) the PIF models.

does not. Although we only prove that re-associations improve lexicographical fairness, simulation results reveal that fairness in terms of balance index also benefits from re-association activities in the proposed game.

For each trial, the difference of the balance indices between the initial and the final association configurations can be viewed as the gain of fairness by re-associations in the trial. Fig. 11(a) displays the average result for the proposed game model. We observe all non-negative gains, with the maximum, average value, and standard deviation 0.558, 0.240 and 0.126, respectively. The gain generally increases with the number of WSs, particularly when adequate APs are provided. The result of the PIF model is shown in Fig. 11(b), where we observe negative to positive fairness gains. The minimal value, average value, and standard deviation of the gain are -0.679, -0.085, and 0.238, respectively. Fig. 11(c) shows excess fairness gains by the proposed game over the PIF model. We found that the superiority of the proposed game over the PIF becomes more significant when fewer APs are introduced. This trend is generally consistent with the behavior exhibited by the expected degree of competition (Fig. 8).

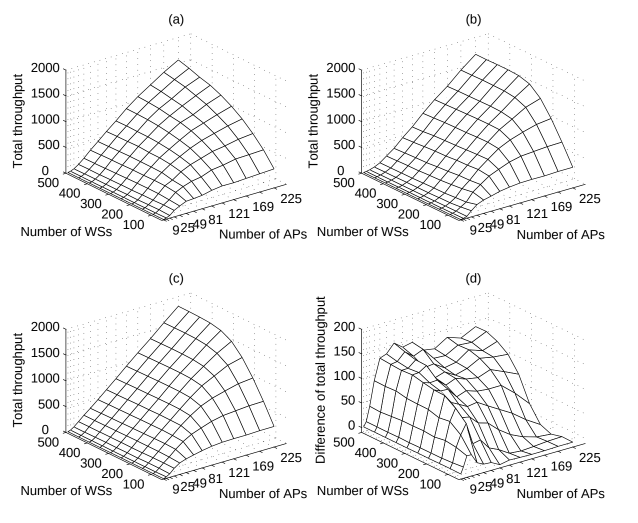
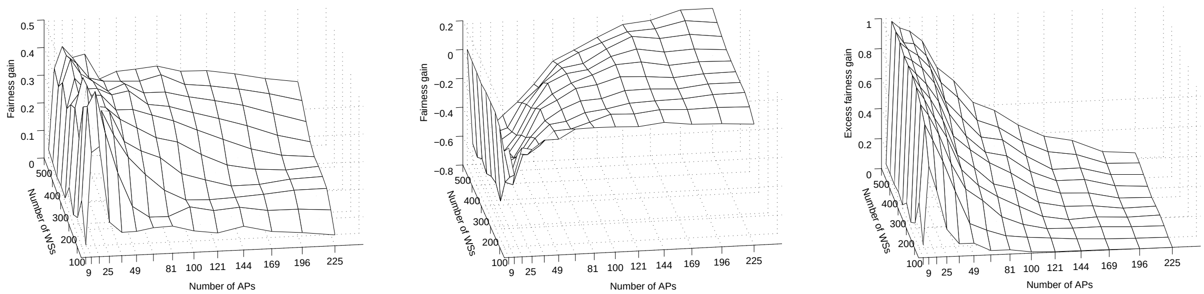


Fig. 12. (a) Aggregated throughputs after the initial RSS-based associations. (b) Aggregated throughputs after the stop of the proposed game model. (c) Aggregated throughputs after the stop of the PIF model. (d) Aggregated throughput of the PIF model minus that of the proposed game model.

C. Aggregated Throughput

Aggregated throughput (counting all WSs) was also investigated. The results of the RSS-based association policy, the proposed game model, and the PIF model are shown in Fig. 12(a)-(c). Clearly, the PIF model outperforms the others. If a configuration is Pareto optimal, then it must have the highest aggregated throughput among all. Therefore, the superiority of the PIF model over our game model further confirms that Nash equilibria in the proposed game are typically not Pareto optimal. Fig. 12(d) shows the excess of the PIF model over



(a) (b) (c)

Fig. 11. Fairness gains by re-associations in (a) the proposed game model and (b) the PIF model. (c) Excess fairness gain by the proposed game over PIF.

the proposed game model in terms of aggregated throughput.

For each trial, the difference of the aggregated throughputs between the initial and the final association configurations is viewed as the gain of aggregated throughput by re-associations in the trial. Fig. 13 shows the gains of aggregated throughput due to re-associations in the proposed game model and in the PIF model. We can see that the PIF model yields all-positive gains while the proposed game model does not necessarily improve aggregated throughput. Fig. 14 compares throughput gains between the proposed game and the PIF models. When few APs are deployed, the difference of the gain between these two models is either negligible (4 APs) or nearly a constant (9 or 16 APs). When more APs are deployed, the throughput gains in both models depend on the number of WSs. For a specific number of APs, there is an optimal number of WSs for which the gain of aggregated throughput due to re-associations is maximized. Deviation from this value diminishes the gain and might even degrade the aggregated throughput (in case of the game model). The optimal number of WSs for 81, 121, 169, and 225 APs are 100, 150, 200, and 250, respectively. For 49 or fewer APs, the optimal number of WSs is smaller than 50. So the results only exhibit a decrease of throughput gain with the number of WSs. This phenomenon can be explained as, when not too many WSs are engaged in a bandwidth competition, the RSS-based association policy fails to fully exploit potential bandwidth collectively offered by all APs, leaving much space for both re-association models to improve. When many WSs are introduced to the access network such that few APs are lightly loaded, the RSS-based association policy leaves little space for re-associations to improve. Thus the throughput gains decline. In particular, a WS in the proposed game is likely to increase its throughput through re-associations at the price of decreasing other WS's throughput. Overall throughput therefore may suffer from such re-associations.

In both models, the maximal gain that can be obtained roughly increases with the number of APs. This is reasonable as more APs provide more potential bandwidth. With 81 or more APs, the proposed game behaves like the PIF model if not too many WSs are involved. Their difference emerges when more WSs are added, and increases with the number of WSs. The PIF model performs better than the game model in finding potential bandwidth the whole access network can provide.

V. Conclusions

This paper has proposed and analyzed an AP selection game where WSs select APs merely to maximize their achievable throughput. We have proven that Nash equilibria exist in such games with the effect of performance anomaly on achievable throughput considered, which guarantees the convergence of configuration transitions. Furthermore, we have shown that association transitions triggered by selfish WSs in fact improve fairness of bandwidth share, which was not expected previously. We conducted extended simulations to study the properties of the proposed game and compared the results with those of the PIF re-association model. The results confirm that the number of association transitions in the proposed game is always limited and generally smaller than that in the PIF model. The proposed game also results in higher bandwidth fairness (in terms of balance index) than the PIF model in all settings. Concerning aggregated throughput, the PIF model always improves the results of the RSS-based association policy, while the proposed game model does not always yield positive improvements.

Acknowledgement

This work has been supported by the National Science Council, Taiwan, under contract NSC 99-2221-E-390-019.

References

[1] L.-H. Yen, T.-T. Yeh, and K.-H. Chi, "Load balancing in IEEE 802.11 networks," *IEEE Internet Computing*, vol. 13, pp. 30–38, Jan. 2009.

[2] Z. Han and K. J. R. Liu, *Resource Allocation for Wireless Networks: Basics, Techniques, and Applications*. Cambridge University Press, 2008.

[3] I. Milchtaich, "Congestion games with player-specfic payoff functions," *Games and Economic Behavior*, vol. 13, no. 1, pp. 111–124, 1996.

[4] M. Cesana, N. Gatti, and I. Malanchini, "Game theoretic analysis of wireless access network selection: models, inefficiency bounds, and algorithms," in *Proc. 3rd Int'l Conf. on Performance Evaluation Methodologies and Tools*, Athens, Greece, Oct. 2008.

[5] M. Cesana, I. Malanchini, and A. Capone, "Modelling network selection and resource allocation in wireless access networks with non-cooperative games," in *The 5th IEEE Int'l Conf. on Mobile Ad Hoc and Sensor Systems*, Atlanta, USA, Oct. 2008, pp. 404–409.

[6] M. Heusse, F. Rousseu, G. Berger-Sabbatel, and A. Duda, "Performance anomaly of 802.11b," in *Proc. IEEE INFOCOM*, Mar. 2003, pp. 836–843.

[7] E. Garcia, D. Viamonte, R. Vidal, and J. Paradells, "Achievable bandwidth estimation for stations in multi-rate IEEE 802.11 WLAN cells," in *Proc. IEEE WoWMoM*, Helsinki, Finland, Jun. 2007, pp. 1–8.

[8] H. Gong and J. Kim, "Dynamic load balancing through association control of mobile users in WiFi networks," *IEEE Trans. on Consumer Electronics*, vol. 54, no. 2, pp. 342–348, May 2008.

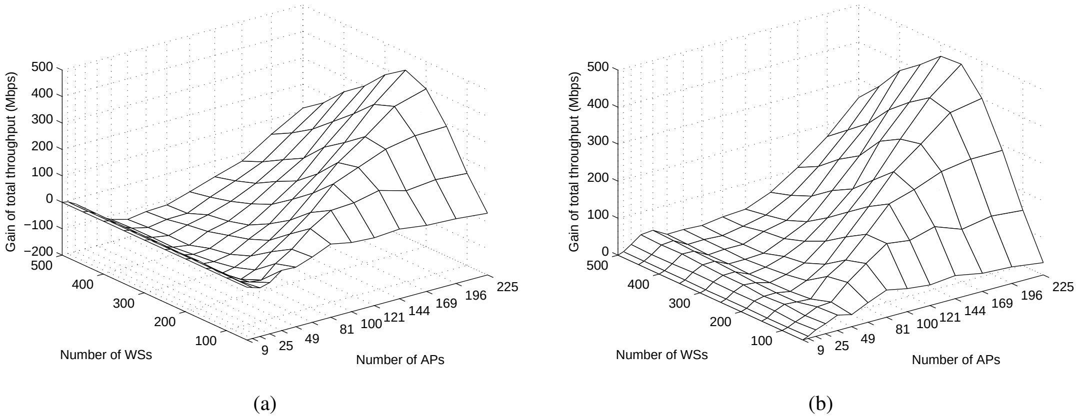


Fig. 13. Gain of aggregated throughput due to re-associations in (a) the proposed game model and (b) the PIF model.

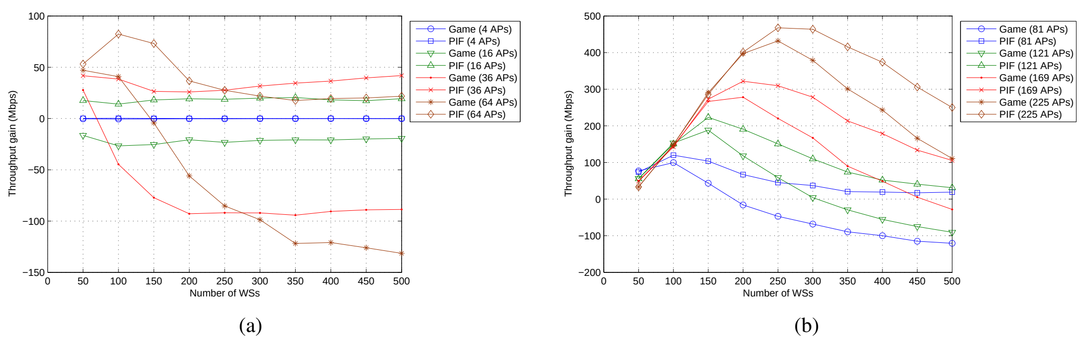


Fig. 14. Comparison of throughput gains between the proposed game and the PIF models model.

[9] Y. Fukuda and Y. Oie, "Decentralized access point selection architecture for wireless LANs: Deployability and robustness," in *Proc. IEEE VTC2004-Fall*, Sep. 2004, pp. 1103–1107.

[10] S. Vasudevan, K. Papagiannaki, C. Diot, J. Kurose, and D. Towsley, "Facilitating access point selection in IEEE 802.11 wireless networks," in *Proc. ACM SIGCOMM/USENIX Internet Measurement Conference*, Berkeley, CA, USA, Oct. 2005.

[11] L.-H. Yen and T.-T. Yeh, "SNMP-based approach to load distribution in IEEE 802.11 networks," in *Proc. IEEE VTC2006-Spring*, Melbourne, Australia, May 2006.

[12] O. Brickley, S. Rea, and D. Pesch, "Load balancing for QoS enhancement in IEEE802.11e WLANs using cell breathing techniques," in *IFIP Mobile and Wireless Communication Networks Conf.*, Marrakech, Morocco, Sep. 2005.

[13] Y. Bejerano, S.-J. Han, and L. Li, "Fairness and load balancing in wireless LANs using association control," in *Proc. ACM MobiCom'04*, Oct. 2004, pp. 315–329.

[14] K. Mittal, E. M. Belding, and S. Suri, "A game-theoretic analysis of wireless access point selection by mobile users," *Computer Communications*, vol. 31, no. 10, pp. 2049–2062, 2008.

[15] S. Shakkottai, E. Altman, and A. Kumar, "Multihoming of users to access points in WLANs: A population game perspective," *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 6, pp. 1207–1215, Aug. 2007.

[16] W. H. Sandholm, *Population Games and Evolutionary Dynamics*. University of Wisconsin, 2006.

[17] R. Chandra, P. Bahl, and P. Bahl, "MultiNet: Connecting to multiple IEEE 802.11 networks using a single wireless card," in *Proc. IEEE INFOCOM 2004*, Hong Kong, Mar. 2004.

[18] L. Jiang, S. Parekh, and J. Walrand, "Base station association game in multi-cell wireless networks," in *Proc. of IEEE WCNC*, 2008.

[19] D. P. Bertsekas and R. Gallager, *Data networks*. New Jersey, USA: Prentice-Hall, 1992.

[20] D.-M. Chiu and R. Jain, "Analysis of the increase and decrease algorithms for congestion avoidance in computer networks," *Computer Networks and ISDN Systems*, vol. 17, pp. 1–14, 1989.

[21] C. L. Liu, *Elements of Discrete Mathematics*, 2nd ed. McGraw-Hill Inc., 1985, p. 145.